Date: 2018-01-13

Turing Machines

Abstract

to be added

Mathematics Subject Classification: 68Q25

Keywords: Turing machine

Comments are welcome. Please send them to john.h.nixon1@gmail.com

These were listed in the order defined before (for ease of verifying the results). By adding the extra arbitrary symbol to these and checking these for reachability according to the algorithm in [2] the following LHS's of IRR(3) with their origins were found (the last part being a length and an equation reference in brackets and can be ignored for now):

$$1b\beta\underline{\delta} \leftarrow 1\underline{\gamma}\epsilon\delta\ 3(44) \quad \begin{array}{l}\text{for } \delta \in \{\mathtt{b},\mathtt{c}\} \text{ and } \beta \in \{\mathtt{a},\mathtt{c},\mathtt{d}\} \\ \text{and } \gamma \in \{\mathtt{d},\mathtt{e}\} \text{ and } \delta = \mathtt{b} \Rightarrow \beta \neq \mathtt{c}\end{array} \tag{1}$$

where

$$\epsilon = \begin{cases} \mathtt{d} \text{ if } \beta = \mathtt{a} \\ \mathtt{a} \text{ if } \beta = \mathtt{c} \\ \mathtt{c} \text{ if } \beta = \mathtt{d} \end{cases} \tag{2}$$

.

$$\left.\begin{array}{l} 2\underline{\gamma}\mathtt{a}\alpha \begin{cases} \overset{\alpha=\mathtt{a}}{\leftarrow} 1\gamma\mathtt{c}\underline{\mathtt{c}}\ 4(50) \\ \overset{\alpha=\mathtt{b}}{\leftarrow} 4\gamma\mathtt{c}\underline{\mathtt{a}}\ 4(73) \\ \overset{\alpha=\mathtt{c}}{\leftarrow} 2\gamma\mathtt{c}\underline{\mathtt{e}}\ 3(45) \\ \overset{\alpha=\mathtt{e}}{\leftarrow} 5\gamma\mathtt{c}\underline{\mathtt{b}}\ 3(48) \end{cases} \\ 2\underline{\gamma}\mathtt{ec} \leftarrow \begin{cases} 3\gamma\mathtt{a}\underline{\mathtt{d}}\ 4^*(66) \\ 4\gamma\mathtt{a}\underline{\mathtt{d}}\ 3(47) \end{cases} \end{array}\right\} \text{for } \gamma \in \{\mathtt{a},\mathtt{c},\mathtt{d}\} \tag{3}$$

$$2\beta\mathtt{b}\underline{\mathtt{e}} \leftarrow 2\underline{\epsilon}\gamma\mathtt{e}\ 4^*(34) \text{ for } \beta \in \{\mathtt{a},\mathtt{c},\mathtt{d}\} \text{ and } \gamma \in \{\mathtt{d},\mathtt{e}\} \tag{4}$$

and ϵ is given by (2)

$$3\underline{\gamma}\beta\alpha \left\{ \begin{array}{l} \overset{\beta=\mathtt{b}}{\longleftarrow} \left\{ \begin{array}{l} \overset{\alpha=\mathtt{a}}{\longleftarrow} 5\gamma\mathtt{a}\underline{\mathtt{d}}\ 4(75) \\ \overset{\alpha=\mathtt{c}}{\longleftarrow} \left\{ \begin{array}{l} 2\gamma\mathtt{a}\underline{\mathtt{b}}\ 3(45) \\ 3\gamma\mathtt{a}\underline{\mathtt{b}}\ 4(63) \end{array} \right. \\ \overset{\alpha=\mathtt{d}}{\longleftarrow} 1\gamma\mathtt{a}\underline{\mathtt{b}}\ 4(49) \end{array} \right. \\ \overset{\beta=\mathtt{c}}{\longleftarrow} \left\{ \begin{array}{l} \overset{\alpha=\mathtt{a}}{\longleftarrow} 3\mathtt{ae}\underline{\mathtt{c}}\ 3(46) \\ \overset{\alpha=\mathtt{d}}{\longleftarrow} 1\mathtt{ae}\underline{\mathtt{a}}\ 3(44) \\ \overset{\alpha=\mathtt{e}}{\longleftarrow} 5\mathtt{ae}\underline{\mathtt{a}}\ 3(48) \end{array} \right. \\ \overset{\beta=\mathtt{e},\alpha=\mathtt{c}}{\longleftarrow} \left\{ \begin{array}{l} 3\gamma\mathtt{b}\underline{\mathtt{d}}\ 4(67) \\ 4\gamma\mathtt{b}\underline{\mathtt{d}}\ 4(74) \end{array} \right. \end{array} \right\} \begin{array}{l} \text{where } \gamma \in \{\mathtt{a},\mathtt{e}\} \text{ and} \\ \text{if } \gamma = \mathtt{e} \text{ then } \beta \neq \mathtt{c} \end{array} \qquad (5)$$

$$\begin{array}{l} 3\mathtt{aa}\underline{\gamma} \leftarrow \left\{ \begin{array}{l} 4\underline{\mathtt{e}}\mathtt{c}\gamma \\ 4\underline{\mathtt{e}}\mathtt{e}\gamma \end{array} \right\} \text{ for } \gamma \in \{\mathtt{b},\mathtt{d}\} \\ 3\alpha\mathtt{b}\underline{\gamma} \left\{ \begin{array}{l} \overset{\alpha=\mathtt{a}}{\longleftarrow} \left\{ \begin{array}{l} 5\underline{\mathtt{c}}\mathtt{e}\gamma\ 3(48) \\ 5\underline{\mathtt{e}}\mathtt{e}\gamma\ 3(48) \end{array} \right. \\ \overset{\alpha=\mathtt{b}}{\longleftarrow} 3\underline{\mathtt{e}}\mathtt{e}\gamma\ 4^*(60) \\ \overset{\alpha=\mathtt{c}}{\longleftarrow} 4\underline{\mathtt{c}}\mathtt{e}\gamma\ 3(47) \end{array} \right\} \gamma \in \{\mathtt{c},\mathtt{d}\} \end{array} \qquad (6)$$

$$4\underline{\delta}\beta\alpha \left\{ \begin{array}{l} \overset{\beta=\mathtt{a},\alpha=\mathtt{c},\delta\in\{\mathtt{c},\mathtt{e}\}}{\longleftarrow} \left\{ \begin{array}{l} 3\delta\mathtt{d}\underline{\mathtt{d}}\ 3(46) \\ 4\delta\mathtt{d}\underline{\mathtt{d}}\ 3(47) \end{array} \right. \\ \overset{\beta=\mathtt{c}}{\longleftarrow} \left\{ \begin{array}{l} \overset{\alpha=\mathtt{a}}{\longleftarrow} \left\{ \begin{array}{l} 3\delta\mathtt{b}\underline{\mathtt{c}}\ 4(65) \\ 1\delta\mathtt{b}\underline{\mathtt{c}}\ 3(44) \end{array} \right. \\ \overset{\alpha=\mathtt{b}}{\longleftarrow} 4\delta\mathtt{b}\underline{\mathtt{a}}\ 4(72) \\ \overset{\alpha=\mathtt{c}}{\longleftarrow} 2\delta\mathtt{b}\underline{\mathtt{e}}\ 4^*(52) \\ \overset{\alpha=\mathtt{d}}{\longleftarrow} 1\delta\mathtt{b}\underline{\mathtt{a}}\ 3(44) \\ \overset{\alpha=\mathtt{e}}{\longleftarrow} \left\{ \begin{array}{l} 5\delta\mathtt{b}\underline{\mathtt{a}}\ 3(48) \\ 5\delta\mathtt{b}\underline{\mathtt{b}}\ 3(48) \end{array} \right. \end{array} \right. \end{array} \right\} \begin{array}{l} \text{for } \delta \in \{\mathtt{b},\mathtt{c},\mathtt{e}\} \text{ and } \beta \in \{\mathtt{a},\mathtt{c}\} \\ \text{and if } \delta = \mathtt{b} \text{ then } \beta = \mathtt{c} \end{array} \qquad (7)$$

$$\left. \begin{array}{l} 4\mathtt{bb}\underline{\beta} \leftarrow 4\underline{\mathtt{b}}\mathtt{b}\beta\ 3(47) \\ 4\mathtt{cb}\underline{\beta} \leftarrow 3\underline{\mathtt{a}}\mathtt{b}\beta\ [\ 4^*(53)\beta = \mathtt{a},\ 4^*(55)\beta = \mathtt{d}] \end{array} \right\} \text{ for } \beta \in \{\mathtt{a},\mathtt{d}\} \qquad (8)$$

$$4\alpha\mathtt{c}\underline{\mathtt{a}} \left\{ \begin{array}{l} \overset{\alpha=\mathtt{a}}{\longleftarrow} \left\{ \begin{array}{l} 5\underline{\mathtt{c}}\mathtt{aa} \\ 5\underline{\mathtt{e}}\mathtt{aa} \end{array} \right. \\ \overset{\alpha=\mathtt{b}}{\longleftarrow} 3\underline{\mathtt{e}}\mathtt{aa}\ 4(61) \\ \overset{\alpha=\mathtt{c}}{\longleftarrow} 4\underline{\mathtt{c}}\mathtt{aa} \end{array} \right. \qquad (9)$$

$$\mathtt{5}\underline{\beta}\mathtt{c}\alpha \left\{ \begin{array}{l} \overset{\alpha=\mathtt{a}}{\leftarrow} \left\{ \begin{array}{l} \mathtt{1}\beta\mathtt{d}\underline{\mathtt{c}}\ 4(51) \\ \mathtt{5}\beta\mathtt{d}\underline{\mathtt{d}}\ 3(48) \end{array}\right. \\ \overset{\alpha=\mathtt{b}}{\leftarrow} \mathtt{4}\beta\mathtt{d}\underline{\mathtt{a}}\ 3(47) \\ \overset{\alpha=\mathtt{c}}{\leftarrow} \left\{ \begin{array}{l} \mathtt{2}\beta\mathtt{d}\underline{\mathtt{b}}\ 3(45) \\ \mathtt{2}\beta\mathtt{d}\underline{\mathtt{e}}\ 3(45) \\ \mathtt{3}\beta\mathtt{d}\underline{\mathtt{b}}\ 3(46) \end{array}\right. \\ \overset{\alpha=\mathtt{d}}{\leftarrow} \mathtt{1}\beta\mathtt{d}\underline{\mathtt{b}}\ 4(51) \\ \overset{\alpha=\mathtt{e}}{\leftarrow} \mathtt{5}\beta\mathtt{d}\underline{\mathtt{b}} \end{array} \right\} \text{for } \beta \in \{\mathtt{c}, \mathtt{e}\} \tag{10}$$

$$\mathtt{5}\alpha\mathtt{a}\underline{\mathtt{d}} \leftarrow \left\{ \begin{array}{l} \overset{\alpha=\mathtt{b}}{\leftarrow} \mathtt{4}\underline{\mathtt{b}}\mathtt{ed}\ 3(47) \\ \overset{\alpha=\mathtt{c}}{\leftarrow} \mathtt{3}\underline{\mathtt{a}}\mathtt{ed}\ 4(58) \end{array}\right. \tag{11}$$

At this point it is clear that this process can be repeated to obtain the LHS's of the IRR(4) from the LHS's of the IRR(3) etc.. Obtaining the LHS's of an IRR(3) rule from the corresponding LHS of the IRR(2) rule involves the use of the AIRR rule indicated after each IRR(2) rule LHS. Also recorded there is the length of the AIRR rule needed and a * if this rule has some RHS's with the pointer at the opposite end from the α. In the same way the AIRR rules to extend IRR(3) rules to the corresponding IRR(4) rules are indicated. This suggests that a great simplification can be made by simply listing the AIRR rules needed to carry this all out (or describe them all recursively) as well as the sets of which AIRR rules follow which by the addition of an extra symbol. This is because it is found that frequently the same AIRR rules are invoked to derive members IRR($\mathtt{n}$) for different values of $\mathtt{n}$.

Reachability testing using a systematic search for all origins can become extremely lengthy and some results can be reused to to make the process more efficient as the following argument shows.

Again, to extend the LHS's of the IRR(3) to the LHS's of the IRR(4), consider each of the IRR(3) of type LR or RL and add each possible single symbol at the opposite end from the pointer such that the resulting CS is reachable. This scheme was applied to generate the

The LHS's of the IRR(4) having state 1 with their origins are as follows

$$\mathtt{1}\alpha\mathtt{b}\beta\underline{\delta} \left\{ \begin{array}{ll} \overset{\alpha=\mathtt{a}}{\leftarrow} \mathtt{2}\underline{\mathtt{d}}\gamma\epsilon\delta\ 5(35) & \text{for } \delta \in \{\mathtt{b}, \mathtt{c}\} \text{ and } \beta \in \{\mathtt{a}, \mathtt{c}, \mathtt{d}\} \\ \overset{\alpha=\mathtt{c}}{\leftarrow} \mathtt{2}\underline{\mathtt{a}}\gamma\epsilon\delta\ 5(35) & \text{except that } \beta \neq \mathtt{c} \text{ if } \delta = \mathtt{b} \\ \overset{\alpha=\mathtt{d}}{\leftarrow} \mathtt{2}\underline{\mathtt{c}}\gamma\epsilon\delta\ 5(35) & \epsilon \text{ is as in (2) and } \gamma \in \{\mathtt{d}, \mathtt{e}\} \end{array}\right. \tag{12}$$

This compact expression resulted from the desire to list the LHS's of the IRR(4) that are in state 1 into the sort order defined by the program (i.e. sort by state then by direction (pointer at left then right in the LHS) then lexicographically by symbols from the pointer (varying most slowly) to the other end of the string). Unfortunately this does not quite work always, but I made it work as closely as possible by grouping the results such that results in one group always sort before results in a later group, and the presentation was optimised as far as I could see.

$$
\begin{array}{l}
\left.\begin{array}{l}
2\underline{\gamma}\mathtt{a}\beta\alpha \left\{\begin{array}{l}
\stackrel{\beta=\mathtt{b}}{\longleftarrow} \left\{\begin{array}{l}
\stackrel{\alpha=\mathtt{a}}{\longleftarrow} \mathtt{5}\gamma\mathtt{ca}\underline{\mathtt{d}}\ 5^*(100)\\
\stackrel{\alpha=\mathtt{c}}{\longleftarrow} \left\{\begin{array}{l} \mathtt{2}\gamma\mathtt{ca}\underline{\mathtt{b}}\ 3(45)\\ \mathtt{3}\gamma\mathtt{ca}\underline{\mathtt{b}}\ 4(63)\end{array}\right.\\
\stackrel{\alpha=\mathtt{d}}{\longleftarrow} \mathtt{1}\gamma\mathtt{ca}\underline{\mathtt{b}}\ 4(49)
\end{array}\right.\\
\stackrel{\beta=\mathtt{c}}{\longleftarrow} \left\{\begin{array}{l}
\stackrel{\alpha=\mathtt{a}}{\longleftarrow} \mathtt{3}\gamma\mathtt{ce}\underline{\mathtt{c}}\ 3(46)\\
\stackrel{\alpha=\mathtt{d}}{\longleftarrow} \mathtt{1}\gamma\mathtt{ce}\underline{\mathtt{a}}\ 3(44)\\
\stackrel{\alpha=\mathtt{e}}{\longleftarrow} \mathtt{5}\gamma\mathtt{ce}\underline{\mathtt{a}}\ 3(48)
\end{array}\right.
\end{array}\right.\\
2\underline{\gamma}\mathtt{ec}\alpha \left\{\begin{array}{l}
\stackrel{\alpha=\mathtt{a}}{\longleftarrow} \left\{\begin{array}{l} \mathtt{1}\gamma\mathtt{ad}\underline{\mathtt{c}}\ 4(51)\\ \mathtt{5}\gamma\mathtt{ad}\underline{\mathtt{d}}\ 3(48)\end{array}\right.\\
\stackrel{\alpha=\mathtt{b}}{\longleftarrow} \mathtt{4}\gamma\mathtt{ad}\underline{\mathtt{a}}\ 3(47)\\
\stackrel{\alpha=\mathtt{c}}{\longleftarrow} \left\{\begin{array}{l} \mathtt{2}\gamma\mathtt{ad}\underline{\mathtt{e}}\ 3(45)\\ \mathtt{2}\gamma\mathtt{ad}\underline{\mathtt{b}}\ 3(45)\\ \mathtt{3}\gamma\mathtt{ad}\underline{\mathtt{b}}\ 3(46)\end{array}\right.\\
\stackrel{\alpha=\mathtt{d}}{\longleftarrow} \mathtt{1}\gamma\mathtt{ad}\underline{\mathtt{b}}\ 4(51)\\
\stackrel{\alpha=\mathtt{e}}{\longleftarrow} \mathtt{5}\gamma\mathtt{ad}\underline{\mathtt{b}}\ 3(48)
\end{array}\right.
\end{array}\right\} \text{for } \gamma \in \{\mathtt{a},\mathtt{c},\mathtt{d}\}\\
2\underline{\mathtt{d}}\mathtt{aec} \leftarrow \left\{\begin{array}{l} \mathtt{3dcb}\underline{\mathtt{d}}\ 5(89)\\ \mathtt{4dcb}\underline{\mathtt{d}}\ 5(98)\end{array}\right.
\end{array}
\tag{13}
$$

The next set of results (14) was obtained from long computations where it is very easy to make a mistake. To make sure that all branches in the search for origins were followed, branches for continuation were marked as such and ticked off when done, and all the results were checked by doing the forward computations.

$$2\alpha\beta\mathtt{b}\underline{\mathtt{e}}\left\{\begin{array}{l} \overset{\beta=\mathtt{a}}{\longleftarrow}\left\{\begin{array}{l}\overset{\alpha=\mathtt{b}}{\longleftarrow}\ \mathtt{1}\underline{\gamma_1}\mathtt{d}\gamma_2\mathtt{e}\ 3(44)\\ \overset{\alpha=\mathtt{c}}{\longleftarrow}\ \mathtt{3}\underline{\mathtt{a}}\mathtt{c}\delta\mathtt{e}\ 4(57)\end{array}\right.\\ \\ \overset{\beta=\mathtt{d}}{\longleftarrow}\left\{\begin{array}{l} \overset{\alpha=\mathtt{a}}{\longleftarrow}\left\{\begin{array}{l} \mathtt{4}\underline{\mathtt{e}}\zeta\mathtt{ce}\ [\zeta=\mathtt{e}\ 3(47)\ \zeta=\mathtt{c}\ 5(95)]\\ \mathtt{5}\underline{\mathtt{c}}\mathtt{ade}\ 3(48)\\ \mathtt{3}\underline{\mathtt{e}}\mathtt{ade}\ 4(61)\end{array}\right.\\ \\ \overset{\alpha=\mathtt{b}}{\longleftarrow}\left\{\begin{array}{l} \mathtt{1}\underline{\gamma_1}\mathtt{c}\gamma_2\mathtt{e}\ 3(44)\\ \mathtt{1}\underline{\gamma_1}\mathtt{abe}\ 3(44)\\ \mathtt{1}\underline{\gamma_1}\mathtt{aae}\ 3(44)\\ \mathtt{4}\underline{\mathtt{b}}\mathtt{cde}\ 5(92)\\ \mathtt{3}\underline{\mathtt{e}}\mathtt{ade}\ 4(61)\end{array}\right.\\ \\ \overset{\alpha=\mathtt{c}}{\longleftarrow}\left\{\begin{array}{l} \mathtt{3}\underline{\mathtt{a}}\mathtt{cde}\ 5(85)\\ \mathtt{4}\underline{\mathtt{c}}\mathtt{ade}\ 5(70)\end{array}\right.\end{array}\right.\end{array}\right\}\ \begin{array}{l}\text{where } \gamma_1 \text{ and } \gamma_2 \text{ are } \in\{\mathtt{d},\mathtt{e}\}\\ \text{and } \delta\in\{\mathtt{b},\mathtt{c}\} \text{ and } \zeta\in\{\mathtt{c},\mathtt{e}\}\end{array} \tag{14}$$

$$
\begin{array}{l}
\left.\begin{array}{l}
\mathtt{3}\underline{\gamma}\mathtt{bac} \leftarrow \begin{cases} \mathtt{3}\gamma\mathtt{ad}\underline{\mathtt{d}}\ 3(46) \\ \mathtt{4}\gamma\mathtt{ad}\underline{\mathtt{d}}\ 3(47) \end{cases} \\
\mathtt{3}\underline{\gamma}\mathtt{bc}\alpha \left\{\begin{array}{l}
\overset{\alpha=\mathtt{a}}{\longleftarrow} \begin{cases} \mathtt{3}\gamma\mathtt{ab}\underline{\mathtt{c}}\ 5(87) \\ \mathtt{1}\gamma\mathtt{ab}\underline{\mathtt{c}}\ 3(44) \end{cases} \\
\overset{\alpha=\mathtt{b}}{\longleftarrow} \mathtt{4}\gamma\mathtt{ab}\underline{\mathtt{a}}\ 4(72) \\
\overset{\alpha=\mathtt{c}}{\longleftarrow} \mathtt{2}\gamma\mathtt{ab}\underline{\mathtt{e}}\ 5^*(83) \\
\overset{\alpha=\mathtt{d}}{\longleftarrow} \mathtt{1}\gamma\mathtt{ab}\underline{\mathtt{a}}\ 3(44) \\
\overset{\alpha=\mathtt{e}}{\longleftarrow} \begin{cases} \mathtt{5}\gamma\mathtt{ab}\underline{\mathtt{a}}\ 3(48) \\ \mathtt{5}\gamma\mathtt{ab}\underline{\mathtt{b}}\ 3(48) \end{cases}
\end{array}\right. \\
\mathtt{3}\underline{\gamma}\mathtt{ec}\alpha \left\{\begin{array}{l}
\overset{\alpha=\mathtt{a}}{\longleftarrow} \begin{cases} \mathtt{1}\gamma\mathtt{bd}\underline{\mathtt{c}}[\gamma=\mathtt{a}\ 5^*(81)\gamma=\mathtt{e}\ 5(82)] \\ \mathtt{5}\gamma\mathtt{bd}\underline{\mathtt{d}}\ 3(48) \end{cases} \\
\overset{\alpha=\mathtt{b}}{\longleftarrow} \mathtt{4}\gamma\mathtt{bd}\underline{\mathtt{a}}\ 3(47) \\
\overset{\alpha=\mathtt{c}}{\longleftarrow} \begin{cases} \mathtt{2}\gamma\mathtt{bd}\underline{\mathtt{e}}\ 3(45) \\ \mathtt{2}\gamma\mathtt{bd}\underline{\mathtt{b}}\ 3(45) \\ \mathtt{3}\gamma\mathtt{bd}\underline{\mathtt{b}}\ 3(46) \end{cases} \\
\overset{\alpha=\mathtt{d}}{\longleftarrow} \mathtt{1}\gamma\mathtt{bd}\underline{\mathtt{b}}[\gamma=\mathtt{a}\ 5^*(77)\gamma=\mathtt{e}\ 5(79)] \\
\overset{\alpha=\mathtt{e}}{\longleftarrow} \mathtt{5}\gamma\mathtt{bd}\underline{\mathtt{b}}\ 3(48)
\end{array}\right.
\end{array}\right\} \text{for } \gamma \in \{\mathtt{a},\mathtt{e}\} \\
\mathtt{3}\underline{\mathtt{a}}\mathtt{ca}\alpha \left\{\begin{array}{l}
\overset{\alpha=\mathtt{a}}{\longleftarrow} \mathtt{1aec}\underline{\mathtt{c}}\ 4(50) \\
\overset{\alpha=\mathtt{b}}{\longleftarrow} \mathtt{4aec}\underline{\mathtt{a}}\ 4(73) \\
\overset{\alpha=\mathtt{c}}{\longleftarrow} \mathtt{2aec}\underline{\mathtt{e}}\ 3(45) \\
\overset{\alpha=\mathtt{e}}{\longleftarrow} \mathtt{5aec}\underline{\mathtt{b}}\ 3(48)
\end{array}\right. \\
\mathtt{3}\underline{\mathtt{a}}\mathtt{cec} \leftarrow \begin{cases} \mathtt{3aea}\underline{\mathtt{d}}\ 4(66) \\ \mathtt{4aea}\underline{\mathtt{d}}\ 3(47) \end{cases}
\end{array} \tag{15}
$$

$$
\begin{array}{l}
\mathtt{3aab}\underline{\mathtt{c}} \leftarrow \begin{cases} \mathtt{4}\underline{\mathtt{e}}\mathtt{cec} \\ \mathtt{4}\underline{\mathtt{e}}\mathtt{eec} \end{cases} \\
\mathtt{3}\alpha\mathtt{bb}\underline{\mathtt{d}} \left\{\begin{array}{l}
\overset{\alpha=\mathtt{a}}{\longleftarrow} \begin{cases} \mathtt{5}\underline{\mathtt{c}}\mathtt{eed}\ 3(48) \\ \mathtt{5}\underline{\mathtt{e}}\mathtt{eed}\ 3(48) \end{cases} \\
\overset{\alpha=\mathtt{b}}{\longleftarrow} \mathtt{3}\underline{\mathtt{e}}\mathtt{eed} \\
\overset{\alpha=\mathtt{c}}{\longleftarrow} \mathtt{4}\underline{\mathtt{c}}\mathtt{eed}\ 3(47)
\end{array}\right. \\
\mathtt{3}\alpha\mathtt{cb}\underline{\mathtt{d}} \left\{\begin{array}{l}
\overset{\alpha=\mathtt{b}}{\longleftarrow} \mathtt{4}\underline{\mathtt{b}}\mathtt{ced}\ 5(93) \\
\overset{\alpha=\mathtt{c}}{\longleftarrow} \mathtt{3}\underline{\mathtt{a}}\mathtt{ced}\ 5(86)
\end{array}\right.
\end{array} \tag{16}
$$

$$
\left.
\begin{array}{l}
4\underline{\delta}\mathtt{ac}\alpha \left\{ \begin{array}{l}
\stackrel{\alpha=\mathtt{a}}{\longleftarrow} \left\{ \begin{array}{l} \mathtt{1}\delta\mathtt{dd}\underline{\mathtt{c}}\ 4(51) \\ \mathtt{5}\delta\mathtt{dd}\underline{\mathtt{d}}\ 3(48) \end{array} \right. \\
\stackrel{\alpha=\mathtt{b}}{\longleftarrow} \mathtt{4}\delta\mathtt{dd}\underline{\mathtt{a}}\ 3(47) \\
\stackrel{\alpha=\mathtt{c}}{\longleftarrow} \left\{ \begin{array}{l} \mathtt{2}\delta\mathtt{dd}\underline{\mathtt{e}}\ 3(45) \\ \mathtt{2}\delta\mathtt{dd}\underline{\mathtt{b}}\ 3(45) \\ \mathtt{3}\delta\mathtt{dd}\underline{\mathtt{b}}\ 3(46) \end{array} \right. \\
\stackrel{\alpha=\mathtt{d}}{\longleftarrow} \mathtt{1}\delta\mathtt{dd}\underline{\mathtt{b}}\ 4(51) \\
\stackrel{\alpha=\mathtt{e}}{\longleftarrow} \mathtt{5}\delta\mathtt{dd}\underline{\mathtt{b}}\ 3(48)
\end{array} \right\} \text{ for } \delta \neq \mathtt{b} \\
4\underline{\delta}\mathtt{ca}\alpha \left\{ \begin{array}{l}
\stackrel{\alpha=\mathtt{a}}{\longleftarrow} \left\{ \begin{array}{l} \mathtt{1}\delta\mathtt{bc}\underline{\mathtt{c}}\ 5(78) \\ \mathtt{3}\delta\mathtt{ee}\underline{\mathtt{c}}\ 3(46) \end{array} \right. \\
\stackrel{\alpha=\mathtt{b}}{\longleftarrow} \mathtt{4}\delta\mathtt{bc}\underline{\mathtt{a}}\ 5^*(96) \\
\stackrel{\alpha=\mathtt{c}}{\longleftarrow} \mathtt{2}\delta\mathtt{bc}\underline{\mathtt{e}}\ 3(45) \\
\stackrel{\alpha=\mathtt{d}}{\longleftarrow} \mathtt{1}\delta\mathtt{ee}\underline{\mathtt{a}}\ 3(44) \\
\stackrel{\alpha=\mathtt{e}}{\longleftarrow} \mathtt{5}\delta\mathtt{bc}\underline{\mathtt{b}}\ 3(48)
\end{array} \right.
\end{array}
\quad
\begin{array}{l}
4\underline{\delta}\mathtt{cb}\alpha \left\{ \begin{array}{l}
\stackrel{\alpha=\mathtt{a}}{\longleftarrow} \mathtt{5}\delta\mathtt{ba}\underline{\mathtt{d}}\ 5^*(99) \\
\stackrel{\alpha=\mathtt{c}}{\longleftarrow} \left\{ \begin{array}{l} \mathtt{2}\delta\mathtt{ba}\underline{\mathtt{b}}\ 3(45) \\ \mathtt{3}\delta\mathtt{ba}\underline{\mathtt{b}}\ 4(63) \\ \mathtt{3}\delta\mathtt{bd}\underline{\mathtt{d}}\ 3(46) \\ \mathtt{4}\delta\mathtt{bd}\underline{\mathtt{d}}\ 3(47) \end{array} \right. \\
\stackrel{\alpha=\mathtt{d}}{\longleftarrow} \mathtt{1}\delta\mathtt{ba}\underline{\mathtt{b}}\ 5^*(76)
\end{array} \right. \\
4\underline{\delta}\mathtt{cc}\alpha \left\{ \begin{array}{l}
\stackrel{\alpha=\mathtt{a}}{\longleftarrow} \mathtt{3}\delta\mathtt{be}\underline{\mathtt{c}}\ 3(46) \\
\stackrel{\alpha=\mathtt{d}}{\longleftarrow} \mathtt{1}\delta\mathtt{be}\underline{\mathtt{a}}\ 3(44) \\
\stackrel{\alpha=\mathtt{e}}{\longleftarrow} \mathtt{5}\delta\mathtt{be}\underline{\mathtt{a}}\ 3(48)
\end{array} \right. \\
4\underline{\delta}\mathtt{cec} \left\{ \begin{array}{l}
\mathtt{3}\delta\mathtt{ba}\underline{\mathtt{d}}\ 4(66) \\
\mathtt{4}\delta\mathtt{ba}\underline{\mathtt{d}}\ 3(47) \\
\mathtt{3}\delta\mathtt{bb}\underline{\mathtt{d}}\ 5^*(88) \\
\mathtt{4}\delta\mathtt{bb}\underline{\mathtt{d}}\ 5^*(97)
\end{array} \right\} \text{ for } \delta \neq \mathtt{c}
\end{array}
\right\} \text{ for } \delta \in \{\mathtt{b},
\tag{17}
$$

$$
\begin{array}{l}
4\alpha\mathtt{bc}\underline{\mathtt{a}} \left\{ \begin{array}{l}
\stackrel{\alpha=\mathtt{a}}{\longleftarrow} \left\{ \begin{array}{l} \mathtt{5}\underline{\mathtt{c}}\mathtt{abd}\ 3(48) \\ \mathtt{5}\underline{\mathtt{e}}\mathtt{abd}\ 3(48) \end{array} \right. \\
\stackrel{\alpha=\mathtt{b}}{\longleftarrow} \mathtt{3}\underline{\mathtt{e}}\mathtt{abd} \\
\stackrel{\alpha=\mathtt{c}}{\longleftarrow} \mathtt{4}\underline{\mathtt{c}}\mathtt{abd}
\end{array} \right. \\
4\alpha\mathtt{bb}\underline{\mathtt{d}} \left\{ \begin{array}{l}
\stackrel{\alpha=\mathtt{b}}{\longleftarrow} \mathtt{4}\underline{\mathtt{b}}\mathtt{bbd}\ 3(47) \\
\stackrel{\alpha=\mathtt{c}}{\longleftarrow} \mathtt{3}\underline{\mathtt{a}}\mathtt{bbd}\ 4(54)
\end{array} \right. \\
4\alpha\mathtt{cb}\underline{\mathtt{d}} \left\{ \begin{array}{l}
\stackrel{\alpha=\mathtt{a}}{\longleftarrow} \left\{ \begin{array}{l} \mathtt{5}\underline{\mathtt{c}}\mathtt{abd}\ 3(48) \\ \mathtt{5}\underline{\mathtt{e}}\mathtt{abd}\ 3(48) \end{array} \right. \\
\stackrel{\alpha=\mathtt{b}}{\longleftarrow} \mathtt{3}\underline{\mathtt{e}}\mathtt{abd}\ 3(61) \\
\stackrel{\alpha=\mathtt{c}}{\longleftarrow} \mathtt{4}\underline{\mathtt{c}}\mathtt{abd}\ 4(70)
\end{array} \right.
\end{array}
\tag{18}
$$

$$\left.\begin{array}{l}
5\underline{\beta}\mathtt{cac} \leftarrow \begin{cases} 3\beta\mathtt{dd}\underline{\mathtt{d}} \\ 4\beta\mathtt{dd}\underline{\mathtt{d}} \end{cases} \\
5\underline{\beta}\mathtt{cb}\alpha \begin{cases} \xleftarrow{\alpha=\mathtt{a}} 5\beta\mathtt{da}\underline{\mathtt{d}}\ 4(75) \\ \xleftarrow{\alpha=\mathtt{c}} \begin{cases} 2\beta\mathtt{da}\underline{\mathtt{b}}\ 3(45) \\ 3\beta\mathtt{da}\underline{\mathtt{b}}\ 4(62)\end{cases} \\ \xleftarrow{\alpha=\mathtt{d}} 1\beta\mathtt{da}\underline{\mathtt{b}}\ 4(49) \end{cases} \\
5\underline{\beta}\mathtt{cc}\alpha \begin{cases} \xleftarrow{\alpha=\mathtt{a}} \begin{cases} 3\beta\mathtt{d}\epsilon\underline{\mathtt{c}}\ 4(64) \\ 1\beta\mathtt{db}\underline{\mathtt{c}}\ 3(44)\end{cases} \\ \xleftarrow{\alpha=\mathtt{b}} 4\beta\mathtt{db}\underline{\mathtt{a}} \\ \xleftarrow{\alpha=\mathtt{c}} 2\beta\mathtt{db}\underline{\mathtt{e}} \\ \xleftarrow{\alpha=\mathtt{d}} 1\beta\mathtt{d}\epsilon\underline{\mathtt{a}}\ 3(44) \\ \xleftarrow{\alpha=\mathtt{e}} \begin{cases} 5\beta\mathtt{d}\epsilon\underline{\mathtt{a}}\ 3(48) \\ 5\beta\mathtt{db}\underline{\mathtt{b}}\ 3(48)\end{cases}\end{cases}
\end{array}\right\} \text{for } \beta\in\{\mathtt{c},\mathtt{e}\} \text{ and } \epsilon\in\{\mathtt{b},\mathtt{e}\} \qquad (19)$$

$$5\alpha\beta\mathtt{a}\underline{\mathtt{d}} \left\{\begin{array}{l}
\xleftarrow{\alpha=\mathtt{a},\beta\neq\mathtt{b}} \begin{cases} 5\underline{\mathtt{c}}\mathtt{aed} \\ 5\underline{\mathtt{e}}\mathtt{aed} \end{cases} \\
\xleftarrow{\alpha=\mathtt{b}} \begin{cases} 4\underline{\mathtt{b}}\mathtt{bed}\ 3(47) \\ 3\underline{\mathtt{e}}\mathtt{aed}\ 4(59) \\ 4\underline{\mathtt{b}}\mathtt{ebd}\ 3(47) \end{cases} \\
\xleftarrow{\alpha=\mathtt{c}} \begin{cases} 3\underline{\mathtt{a}}\mathtt{bed} \\ 4\underline{\mathtt{c}}\mathtt{aed} \\ 3\underline{\mathtt{a}}\mathtt{ebd} \end{cases}
\end{array}\right\} \text{for } \beta\in\{\mathtt{b},\mathtt{c}\} \qquad (20)$$

****

1 Continuing the analysis of the TM

In order to extend equation (12) by one symbol, first forward computation shows that a rule of type RL will only result from forward computation of the LHS of (12) if $\alpha \in \{\mathtt{a},\mathtt{d}\}$ when $\beta \in \{\mathtt{a},\mathtt{c}\}$ and $\alpha \in \{\mathtt{c},\mathtt{d}\}$ when $\beta = \mathtt{d}$. The first step according to the above scheme for doing the reverse computation search gives

$$1\alpha\alpha_1\mathtt{b}\beta\underline{\delta} \leftarrow \begin{cases} \xleftarrow{\alpha_1=\mathtt{a}} 2\alpha\underline{\mathtt{d}}\gamma\epsilon\delta & \text{for } \delta\in\{\mathtt{b},\mathtt{c}\}, \beta\in\{\mathtt{a},\mathtt{c},\mathtt{d}\} \\ \xleftarrow{\alpha_1=\mathtt{c}} 2\alpha\underline{\mathtt{a}}\gamma\epsilon\delta & \beta\neq\mathtt{c} \text{ if } \delta=\mathtt{b}, \gamma\in\{\mathtt{d},\mathtt{e}\} \\ \xleftarrow{\alpha_1=\mathtt{d}} 2\alpha\underline{\mathtt{c}}\gamma\epsilon\delta & \alpha_1\in\{\mathtt{a},\mathtt{d}\} \text{ if } \beta\in\{\mathtt{a},\mathtt{c}\} \text{ and } \alpha_1\in\{\mathtt{c},\mathtt{d}\} \text{ if } \beta=\mathtt{d} \end{cases} \qquad (21)$$

Then, treating all three branches together, the auxiliary reverse rule is derived as follows starting with the truncated one of length 3:

$$2\alpha\underline{\zeta}\gamma \begin{cases} \overset{\alpha=\mathtt{b}}{\longleftarrow} \begin{cases} \mathtt{1}\underline{\mathtt{d}}\zeta\gamma \\ \mathtt{1}\underline{\mathtt{e}}\zeta\gamma \end{cases} \\ \overset{\gamma=\mathtt{d}}{\longleftarrow} \mathtt{1}\alpha\zeta\underline{\mathtt{a}} \\ \overset{\gamma=\mathtt{e}}{\longleftarrow} \mathtt{5}\alpha\zeta\underline{\mathtt{a}} \end{cases} \qquad \text{for } \zeta \in \{\mathtt{a},\mathtt{c},\mathtt{d}\} \text{ and } \gamma \in \{\mathtt{d},\mathtt{e}\} \tag{22}$$

Then adding back one more symbol and tracing all the branches to their origins gives the tree that can be represented by the results (23)-(33) (remembering the restriction (2) on ϵ):

$$2\alpha\underline{\zeta}\gamma\epsilon \begin{cases} \overset{\alpha=\mathtt{b}}{\longleftarrow} \begin{cases} \mathtt{1}\underline{\mathtt{d}}\zeta\gamma\epsilon \\ \mathtt{1}\underline{\mathtt{e}}\zeta\gamma\epsilon \end{cases} \\ \overset{\gamma=\mathtt{d}}{\longleftarrow} \mathtt{1}\alpha\zeta\underline{\mathtt{a}}\epsilon \leftarrow 2\alpha\underline{\epsilon_1}\mathtt{a}\epsilon \begin{cases} \overset{\alpha=\mathtt{b}}{\longleftarrow} \mathtt{1}\underline{\gamma_1}\epsilon_1\mathtt{a}\epsilon \\ \leftarrow 3\alpha\epsilon_1\underline{\mathtt{c}}\epsilon \end{cases} \\ \overset{\gamma=\mathtt{e}}{\longleftarrow} \mathtt{5}\alpha\zeta\underline{\mathtt{a}}\epsilon \end{cases} . \tag{23}$$

Here ϵ and ζ are $\in \{\mathtt{a},\mathtt{c},\mathtt{d}\}$, $\gamma \in \{\mathtt{d},\mathtt{e}\}$, and $\gamma_1 \in \{\mathtt{d},\mathtt{e}\}$ as is γ. ϵ_1 is related to ζ in the same way that ϵ is related to β in (2) which gives

α_1	ζ	ϵ_1
a	d	c
c	a	d
d	c	a

(24)

$$3\alpha\epsilon_1\underline{\mathtt{c}}\epsilon \begin{cases} \overset{\epsilon\neq\mathtt{d}}{\longleftarrow} \mathtt{n}_2\alpha\mathtt{dc}\underline{\eta} \\ \overset{\epsilon_1=\mathtt{a}}{\longleftarrow} \begin{cases} \mathtt{5}\alpha\underline{\mathtt{c}}\mathtt{c}\epsilon \begin{cases} \overset{\alpha=\mathtt{a}}{\longleftarrow} \mathtt{4}\underline{\mathtt{e}}\mathtt{cc}\epsilon \\ \leftarrow \mathtt{3}\alpha\mathtt{c}\underline{\mathtt{d}}\epsilon \\ \leftarrow \mathtt{4}\alpha\mathtt{c}\underline{\mathtt{d}}\epsilon \end{cases} \\ \mathtt{5}\alpha\underline{\mathtt{e}}\mathtt{c}\epsilon \end{cases} \\ \overset{\epsilon_1=\mathtt{c}}{\longleftarrow} \mathtt{4}\alpha\underline{\mathtt{c}}\mathtt{c}\epsilon \end{cases} \tag{25}$$

where $\mathtt{n}_2$ and η are functions of ϵ defined by

ϵ	$\mathtt{n}_2$	η
a	1	c
b	4	a
c	2	e
e	5	b

(26)

which when combined with (2) gives

β	ϵ	$\mathtt{n}_2$	η
a	d		
c	a	1	c
d	c	2	e

. (27)

$$\mathtt{3}\alpha\mathtt{c\underline{d}}\epsilon \leftarrow \left\{\begin{array}{l} \leftarrow \mathtt{4}\alpha\mathtt{\underline{c}d}\epsilon \left\{\begin{array}{l} \leftarrow \mathtt{1}\alpha\mathtt{c\underline{b}}\epsilon \\ \overset{\alpha=\mathtt{b}}{\leftarrow} \mathtt{4\underline{b}cd}\epsilon \\ \overset{\alpha=\mathtt{c}}{\leftarrow} \mathtt{3\underline{a}cd}\epsilon \end{array}\right. \\ \overset{\epsilon=\mathtt{a}}{\leftarrow} \mathtt{1}\alpha\mathtt{cd\underline{c}} \\ \overset{\epsilon=\mathtt{c}}{\leftarrow} \mathtt{2}\alpha\mathtt{cd\underline{e}} \end{array}\right. \tag{28}$$

$$\mathtt{1}\alpha\mathtt{c\underline{b}}\epsilon \leftarrow \mathtt{2}\alpha\mathtt{\underline{a}b}\epsilon \overset{\alpha=\mathtt{b}}{\leftarrow} \left\{\begin{array}{l} \mathtt{1\underline{d}ab}\epsilon \\ \mathtt{1\underline{e}ab}\epsilon \end{array}\right. \tag{29}$$

$$\mathtt{4}\alpha\mathtt{c\underline{d}}\epsilon \left\{\begin{array}{l} \leftarrow \mathtt{3}\alpha\mathtt{\underline{a}d}\epsilon \left\{\begin{array}{l} \overset{\alpha=\mathtt{a}}{\leftarrow} \left\{\begin{array}{l} \mathtt{5\underline{c}ad}\epsilon \\ \mathtt{5\underline{e}ad}\epsilon \end{array}\right. \\ \overset{\alpha=\mathtt{b}}{\leftarrow} \mathtt{3\underline{e}ad}\epsilon \\ \overset{\alpha=\mathtt{c}}{\leftarrow} \mathtt{4\underline{c}ad}\epsilon \end{array}\right. \\ \overset{\epsilon=\mathtt{a}}{\leftarrow} \mathtt{5}\alpha\mathtt{cd\underline{d}} \\ \overset{\epsilon=\mathtt{c}}{\leftarrow} \left\{\begin{array}{l} \mathtt{2}\alpha\mathtt{cd\underline{b}} \\ \mathtt{3}\alpha\mathtt{cd\underline{b}} \end{array}\right. \\ \overset{\epsilon=\mathtt{d}}{\leftarrow} \mathtt{1}\alpha\mathtt{cd\underline{b}} \end{array}\right. \tag{30}$$

$$\mathtt{5}\alpha\mathtt{\underline{e}c}\epsilon \left\{\begin{array}{l} \overset{\alpha=\mathtt{a}}{\leftarrow} \mathtt{4\underline{e}ec}\epsilon \\ \leftarrow \mathtt{3}\alpha\mathtt{e\underline{d}}\epsilon \left\{\begin{array}{l} \overset{\epsilon=\mathtt{a}}{\leftarrow} \mathtt{1}\alpha\mathtt{ed\underline{c}} \\ \overset{\epsilon=\mathtt{c}}{\leftarrow} \mathtt{2}\alpha\mathtt{ed\underline{e}} \end{array}\right. \\ \leftarrow \mathtt{4}\alpha\mathtt{e\underline{d}}\epsilon \left\{\begin{array}{l} \overset{\epsilon=\mathtt{a}}{\leftarrow} \mathtt{5}\alpha\mathtt{ed\underline{d}} \\ \overset{\epsilon=\mathtt{c}}{\leftarrow} \left\{\begin{array}{l} \mathtt{2}\alpha\mathtt{ed\underline{b}} \\ \mathtt{3}\alpha\mathtt{ed\underline{b}} \end{array}\right. \\ \overset{\epsilon=\mathtt{d}}{\leftarrow} \mathtt{1}\alpha\mathtt{ed\underline{b}} \end{array}\right. \end{array}\right. \tag{31}$$

$$\mathtt{4}\alpha\mathtt{\underline{c}c}\epsilon \left\{\begin{array}{l} \overset{\alpha=\mathtt{b}}{\leftarrow} \mathtt{4\underline{b}cc}\epsilon \\ \overset{\alpha=\mathtt{c}}{\leftarrow} \mathtt{3\underline{a}cc}\epsilon \\ \leftarrow \mathtt{2}\alpha\mathtt{c\underline{b}}\epsilon \left\{\begin{array}{l} \overset{\epsilon=\mathtt{a}}{\leftarrow} \mathtt{3}\alpha\mathtt{cb\underline{c}} \\ \overset{\epsilon=\mathtt{d}}{\leftarrow} \mathtt{1}\alpha\mathtt{cb\underline{a}} \end{array}\right. \\ \leftarrow \mathtt{3}\alpha\mathtt{c\underline{b}}\epsilon \left\{\begin{array}{l} \leftarrow \mathtt{4}\alpha\mathtt{\underline{c}b}\epsilon \left\{\begin{array}{l} \overset{\alpha=\mathtt{b}}{\leftarrow} \mathtt{4\underline{b}cb}\epsilon \\ \overset{\alpha=\mathtt{c}}{\leftarrow} \mathtt{3\underline{a}cb}\epsilon \end{array}\right. \\ \overset{\epsilon=\mathtt{a}}{\leftarrow} \mathtt{1}\alpha\mathtt{cb\underline{c}} \\ \overset{\epsilon=\mathtt{c}}{\leftarrow} \mathtt{2}\alpha\mathtt{cb\underline{e}} \end{array}\right. \end{array}\right. \tag{32}$$

$$\mathtt{5}\alpha\zeta\mathtt{\underline{a}}\epsilon \left\{\begin{array}{l} \overset{\zeta=\mathtt{a}}{\leftarrow} \mathtt{4}\alpha\mathtt{\underline{e}a}\epsilon \left\{\begin{array}{l} \overset{\alpha=\mathtt{b}}{\leftarrow} \mathtt{4\underline{b}ea}\epsilon \\ \overset{\alpha=\mathtt{c}}{\leftarrow} \mathtt{3\underline{a}ea}\epsilon \\ \leftarrow \mathtt{5}\alpha\mathtt{e\underline{d}}\epsilon \overset{\epsilon=\mathtt{c}}{\leftarrow} \left\{\begin{array}{l} \mathtt{3}\alpha\mathtt{ed\underline{d}} \\ \mathtt{4}\alpha\mathtt{ed\underline{d}} \end{array}\right. \end{array}\right. \\ \overset{\epsilon=\mathtt{c}}{\leftarrow} \left\{\begin{array}{l} \mathtt{3}\alpha\zeta\mathtt{a\underline{d}} \\ \mathtt{4}\alpha\zeta\mathtt{a\underline{d}} \end{array}\right. \end{array}\right. \tag{33}$$

The equations (23)-(33) can be summarised the following single tree:

$$
2\alpha\underline{\zeta}\gamma\epsilon
\left\{
\begin{array}{l}
\stackrel{\gamma=\mathtt{d},\epsilon\neq\mathtt{d}}{\longleftarrow} \mathtt{n}_2\alpha\mathtt{dc}\underline{\eta} \\
\stackrel{\gamma=\mathtt{e},\epsilon=\mathtt{c}}{\longleftarrow} \left\{\ 3\alpha\zeta\mathtt{a}\underline{\mathtt{d}}\ \ 4\alpha\zeta\mathtt{a}\underline{\mathtt{d}}\ \right\} \\
\stackrel{\zeta=\mathtt{c},\gamma=\mathtt{d}}{\longleftarrow}
\left\{
\begin{array}{l}
\stackrel{\epsilon=\mathtt{a}}{\longleftarrow} \left\{\ 1\alpha\mathtt{cd}\underline{\mathtt{c}}\ \ 5\alpha\mathtt{cd}\underline{\mathtt{d}}\ \ 1\alpha\mathtt{ed}\underline{\mathtt{c}}\ \ 5\alpha\mathtt{ed}\underline{\mathtt{d}}\ \right\} \\
\stackrel{\epsilon=\mathtt{c}}{\longleftarrow} \left\{
\begin{array}{lll}
2\alpha\mathtt{cd}\underline{\mathtt{e}} & 2\alpha\mathtt{cd}\underline{\mathtt{b}} & 3\alpha\mathtt{cd}\underline{\mathtt{b}} \\
2\alpha\mathtt{ed}\underline{\mathtt{e}} & 2\alpha\mathtt{ed}\underline{\mathtt{b}} & 3\alpha\mathtt{ed}\underline{\mathtt{b}}
\end{array}
\right\} \\
\stackrel{\epsilon=\mathtt{d}}{\longleftarrow} \left\{\ 1\alpha\mathtt{cd}\underline{\mathtt{b}}\ \ 1\alpha\mathtt{ed}\underline{\mathtt{b}}\ \right\}
\end{array}
\right. \\
\stackrel{\zeta=\mathtt{d},\gamma=\mathtt{d}}{\longleftarrow}
\left\{
\begin{array}{l}
\stackrel{\epsilon=\mathtt{a}}{\longleftarrow} \left\{\ 3\alpha\mathtt{cb}\underline{\mathtt{c}}\ \ 1\alpha\mathtt{cb}\underline{\mathtt{c}}\ \right\} \\
\stackrel{\epsilon=\mathtt{c}}{\longleftarrow} 2\alpha\mathtt{cb}\underline{\mathtt{e}} \\
\stackrel{\epsilon=\mathtt{d}}{\longleftarrow} 1\alpha\mathtt{cb}\underline{\mathtt{a}}
\end{array}
\right. \\
\stackrel{\zeta=\mathtt{a},\gamma=\mathtt{e},\epsilon=\mathtt{c}}{\longleftarrow} \left\{\ 3\alpha\mathtt{ed}\underline{\mathtt{d}}\ \ 4\alpha\mathtt{ed}\underline{\mathtt{d}}\ \right\} \\
\stackrel{\alpha=\mathtt{a},\zeta=\mathtt{c},\gamma=\mathtt{d}}{\longleftarrow} \left\{\ 4\underline{\mathtt{e}}\mathtt{cc}\epsilon\ \ 5\underline{\mathtt{c}}\mathtt{ad}\epsilon\ \ 5\underline{\mathtt{e}}\mathtt{ad}\epsilon\ \ 4\underline{\mathtt{e}}\mathtt{ec}\epsilon\ \right\} \\
\stackrel{\alpha=\mathtt{b}}{\longleftarrow}
\left\{
\begin{array}{l}
\longleftarrow \left\{\ 1\underline{\mathtt{d}}\zeta\gamma\epsilon\ \ 1\underline{\mathtt{e}}\zeta\gamma\epsilon\ \right\} \\
\stackrel{\gamma=\mathtt{d}}{\longleftarrow} 1\underline{\gamma_1}\epsilon_1\mathtt{a}\epsilon \\
\stackrel{\zeta=\mathtt{c},\gamma=\mathtt{d}}{\longleftarrow} \left\{\ 4\underline{\mathtt{b}}\mathtt{cd}\epsilon\ \ 1\underline{\mathtt{d}}\mathtt{ab}\epsilon\ \ 1\underline{\mathtt{e}}\mathtt{ab}\epsilon\ \ 3\underline{\mathtt{e}}\mathtt{ad}\epsilon\ \right\} \\
\stackrel{\zeta=\mathtt{d},\gamma=\mathtt{d}}{\longleftarrow} \left\{\ 4\underline{\mathtt{b}}\mathtt{cc}\epsilon\ \ 4\underline{\mathtt{b}}\mathtt{cb}\epsilon\ \right\} \\
\stackrel{\zeta=\mathtt{a},\gamma=\mathtt{e}}{\longleftarrow} 4\underline{\mathtt{b}}\mathtt{ea}\epsilon
\end{array}
\right. \\
\stackrel{\alpha=\mathtt{c}}{\longleftarrow}
\left\{
\begin{array}{l}
\stackrel{\zeta=\mathtt{c},\gamma=\mathtt{d}}{\longleftarrow} \left\{\ 4\underline{\mathtt{c}}\mathtt{ad}\epsilon\ \ 3\underline{\mathtt{a}}\mathtt{dc}\epsilon\ \right\} \\
\stackrel{\zeta=\mathtt{d},\gamma=\mathtt{d}}{\longleftarrow} \left\{\ 3\underline{\mathtt{a}}\mathtt{cc}\epsilon\ \ 3\underline{\mathtt{a}}\mathtt{cb}\epsilon\ \right\} \\
\stackrel{\zeta=\mathtt{a},\gamma=\mathtt{e}}{\longleftarrow} 3\underline{\mathtt{a}}\mathtt{ea}\epsilon
\end{array}
\right.
\end{array}
\right. \qquad * \tag{34}
$$

The order of the place holder symbols was chosen as $\alpha\zeta\gamma\epsilon$ and the skeleton result showing all the branching was developed, and the numbers of results on each branch, then all the results were put in resulting in (34). This result due to its complexity would be very difficult to obtain without being extremely systematic. Also it does not naturally split into special cases. It can of course be done, but this would result in some repetition. This is because most of the parameters are not mentioned in most branches. Only γ which is mentioned in all but one branch so splitting by this would give minimal repetition. ζ is mentioned in all but 4 branches.

In equation (34), ζ and ϵ are in $\{\mathtt{a},\mathtt{c},\mathtt{d}\}$ and $\gamma \in \{\mathtt{d},\mathtt{e}\}$. Now a $\delta \in \{\mathtt{b},\mathtt{c}\}$ can be added on the right hand end of the strings and $\epsilon \neq \mathtt{a}$ if $\delta = \mathtt{b}$ follows from (12) and (2), then the backward searching continues resulting in (35) following.

This can be done case by case for each of the 10 cases in (21) and using (34). To simplify this a little, note that the 3 main bottom branches (for $\alpha \in \{\mathtt{a},\mathtt{b},\mathtt{c}\}$) always terminate with the pointer at the left for each case. For each $\alpha \in \{\mathtt{b},\mathtt{c}\}$ for each $\zeta \in \{\mathtt{a},\mathtt{c},\mathtt{d}\}$ such results are found and if $\alpha = \mathtt{a}$ the same is true for only $\zeta = \texttt{c}$. Whatever symbol δ is added on the right in (21), when followed though to (34), these reverse computations still terminate there with the pointer at the left and demonstrate reachability of the LHS of (21) for $\alpha \in \{\texttt{b}, \texttt{c}\}$ and for $\alpha = \texttt{a}$ if $\zeta = \texttt{c}$ i.e. $\alpha_{\texttt{1}} = \texttt{d}$. Calculation shows that these are the only cases. This follows by carrying out the backward search algorithm starting from the terminating CS's on each of the first 5 main branches of (34) to which a single c or d (δ) has been added at the right and showing that the final results always have the pointer at the right, thus not establishing any more conditions for the reachability of the LHS of (21).

$$
\mathtt{1}\alpha\alpha_1\mathtt{b}\beta\underline{\delta} \leftarrow \mathtt{2}\alpha\underline{\zeta}\gamma\epsilon\delta
\begin{cases}
\xleftarrow{\gamma=\mathtt{e},\epsilon=\mathtt{c}} \begin{cases} \xleftarrow{\delta=\mathtt{b}} \mathtt{4}\alpha\zeta\mathtt{ad\underline{a}} \\ \xleftarrow{\delta=\mathtt{c}} \left\{ \mathtt{2}\alpha\zeta\mathtt{ad\underline{e}} \quad \mathtt{2}\alpha\zeta\mathtt{ad\underline{b}} \quad \mathtt{3}\alpha\zeta\mathtt{ad\underline{b}} \right\} \end{cases} \\
\xleftarrow{\zeta=\mathtt{c},\gamma=\mathtt{d}} \begin{cases} \xleftarrow{\epsilon=\mathtt{a},\delta=\mathtt{c}} \left\{ \mathtt{3}\alpha\mathtt{cdd\underline{d}} \quad \mathtt{4}\alpha\mathtt{cdd\underline{d}} \quad \mathtt{3}\alpha\mathtt{edd\underline{d}} \quad \mathtt{4}\alpha\mathtt{edd\underline{d}} \right\} \\ \xleftarrow{\epsilon=\mathtt{c}} \begin{cases} \xleftarrow{\delta=\mathtt{b}} \left\{ \mathtt{4}\alpha\mathtt{cdb\underline{a}} \quad \mathtt{4}\alpha\mathtt{edb\underline{a}} \right\} \\ \xleftarrow{\delta=\mathtt{c}} \left\{ \mathtt{2}\alpha\mathtt{cdb\underline{e}} \quad \mathtt{2}\alpha\mathtt{edb\underline{e}} \right\} \end{cases} \end{cases} \\
\xleftarrow{\zeta=\mathtt{d},\gamma=\mathtt{d}} \begin{cases} \xleftarrow{\epsilon=\mathtt{a}} \begin{cases} \xleftarrow{\delta=\mathtt{b}} \mathtt{4}\alpha\mathtt{cbc\underline{a}} \\ \xleftarrow{\delta=\mathtt{c}} \mathtt{2}\alpha\mathtt{cbc\underline{e}} \end{cases} \\ \xleftarrow{\epsilon=\mathtt{c},\delta=\mathtt{c}} \left\{ \mathtt{3}\alpha\mathtt{dcb\underline{d}} \quad \mathtt{4}\alpha\mathtt{dcb\underline{d}} \right\} \end{cases} \\
\xleftarrow{\zeta=\mathtt{a},\gamma=\mathtt{e},\epsilon=\mathtt{c}} \begin{cases} \xleftarrow{\delta=\mathtt{c}} \left\{ \mathtt{2}\alpha\mathtt{edd\underline{e}} \quad \mathtt{2}\alpha\mathtt{edd\underline{b}} \quad \mathtt{3}\alpha\mathtt{edd\underline{b}} \right\} \\ \xleftarrow{\delta=\mathtt{b}} \mathtt{4}\alpha\mathtt{edd\underline{a}} \end{cases} \\
\xleftarrow{\alpha=\mathtt{a},\zeta=\mathtt{c},\gamma=\mathtt{d}} \left\{ \mathtt{4\underline{e}cc}\epsilon\delta\ 6^*(102) \quad \mathtt{5\underline{c}ad}\epsilon\delta\ 3(48) \quad \mathtt{5\underline{e}ad}\epsilon\delta\ 3(48) \quad \mathtt{4\underline{e}ec}\epsilon\delta\ 3(47) \right\} \\
\xleftarrow{\alpha=\mathtt{b}} \begin{cases} \leftarrow \left\{ \mathtt{1\underline{d}}\zeta\gamma\epsilon\delta\ 3(44) \quad \mathtt{1\underline{e}}\zeta\gamma\epsilon\delta\ 3(44) \right\} \\ \xleftarrow{\gamma=\mathtt{d}} \mathtt{1}\underline{\gamma_1}\epsilon_1\mathtt{a}\epsilon\delta\ 3(44) \\ \xleftarrow{\zeta=\mathtt{a},\gamma=\mathtt{e}} \mathtt{4\underline{b}ea}\epsilon\delta\ 3(47) \\ \xleftarrow{\zeta=\mathtt{c},\gamma=\mathtt{d}} \left\{ \mathtt{4\underline{b}cd}\epsilon\delta\ 5(92) \quad \mathtt{1\underline{d}ab}\epsilon\delta\ 3(44) \quad \mathtt{1\underline{e}ab}\epsilon\delta\ 3(44) \quad \mathtt{3\underline{e}ad}\epsilon\delta\ 4(59) \right\} \\ \xleftarrow{\zeta=\mathtt{d},\gamma=\mathtt{d}} \begin{cases} \mathtt{4\underline{b}cc}\epsilon\delta \text{ see } 4^*(69) \text{ and } 5^*(91) \\ \mathtt{4\underline{b}cb}\epsilon\delta \text{ see } 4^*(68) \\ \xleftarrow{\epsilon=\mathtt{a}} \mathtt{4\underline{b}cec}\delta \\ \xleftarrow{\epsilon=\mathtt{c}} \begin{cases} \mathtt{1\underline{d}ade}\delta \quad \mathtt{1\underline{e}ade}\delta \quad \mathtt{1\underline{d}aee}\delta \quad \mathtt{1\underline{e}aee}\delta \\ \mathtt{1\underline{d}dae}\delta \quad \mathtt{1\underline{e}dae}\delta \quad \mathtt{4\underline{b}eae}\delta \end{cases} \end{cases} \\ \xleftarrow{\gamma=\mathtt{e},\epsilon=\mathtt{c}} \left\{ \mathtt{4\underline{b}ecd}\delta \quad \mathtt{4\underline{b}eed}\delta \right\} \end{cases} \\
\xleftarrow{\alpha=\mathtt{c}} \begin{cases} \xleftarrow{\zeta=\mathtt{a},\gamma=\mathtt{e}} \mathtt{3\underline{a}ea}\epsilon\delta \\ \xleftarrow{\zeta=\mathtt{c},\gamma=\mathtt{d}} \left\{ \mathtt{4\underline{c}ad}\epsilon\delta \quad \mathtt{3\underline{a}dc}\epsilon\delta \right\} \\ \xleftarrow{\zeta=\mathtt{d},\gamma=\mathtt{d}} \begin{cases} \mathtt{3\underline{a}cc}\epsilon\delta \\ \mathtt{3\underline{a}cb}\epsilon\delta \\ \xleftarrow{\epsilon=\mathtt{a}} \mathtt{3\underline{a}cec}\delta \\ \xleftarrow{\epsilon=\mathtt{c}} \mathtt{3\underline{a}eae}\delta \end{cases} \\ \xleftarrow{\gamma=\mathtt{e},\epsilon=\mathtt{c}} \left\{ \mathtt{3\underline{a}ecd}\delta \quad \mathtt{3\underline{a}eed}\delta \right\} \end{cases}
\end{cases}
\tag{35}
$$

Next, similar arguments should be followed starting from (13) to obtain all the LHS's of the IRR(5) together with their origins.

The presentation of the above results might be improved by treating more of the cases separately rather than all combined.

For the first set in equation (13), all the 18 LHS's give rise to rules of type LR under forward computation. Replacing α by α_1, and adding α on the right provides the starting point for continuing the backward search algorithm to find origins. The auxiliary reverse rules found in each case are added to the list (at the end of the paper) using the minimum number of symbols. This is analogous to the irreducible rules for forward computation. An asterisk(*) will be added to indicate that reverse rule needs to be extended to an irreducible reverse rule that is longer. This happens when an origin is found that has the pointer at the opposite end from the α. The resulting set of LHS's of IRR(5) and their origins corresponding to the first part of (13) are given next.

Therefore the following is the set of LHS's of IRR(5) with their origins that correspond to the first part of(13):

$$2\underline{\gamma}\mathtt{a}\beta\alpha_1\alpha \left\{ \begin{array}{ll} \stackrel{\beta=\mathtt{b},\alpha_1=\mathtt{c}}{\longleftarrow} & \left\{ \begin{array}{l} \stackrel{\alpha=\mathtt{a}}{\longleftarrow} \left\{\ 3\gamma\mathtt{cab\underline{c}} \quad 1\gamma\mathtt{cab\underline{c}}\ \right\} \\ \stackrel{\alpha=\mathtt{b}}{\longleftarrow} 4\gamma\mathtt{cab\underline{a}} \\ \stackrel{\alpha=\mathtt{c}}{\longleftarrow} 2\gamma\mathtt{cab\underline{e}} \\ \stackrel{\alpha=\mathtt{d}}{\longleftarrow} 1\gamma\mathtt{cab\underline{a}} \\ \stackrel{\alpha=\mathtt{e}}{\longleftarrow} \left\{\ 5\gamma\mathtt{cab\underline{a}} \quad 5\gamma\mathtt{cab\underline{b}}\ \right\} \end{array} \right. \\ \stackrel{\beta=\mathtt{c}}{\longleftarrow} & \left\{ \begin{array}{ll} \stackrel{\alpha_1=\mathtt{a}}{\longleftarrow} & \left\{ \begin{array}{l} \stackrel{\alpha=\mathtt{a}}{\longleftarrow} 1\gamma\mathtt{cec\underline{c}} \\ \stackrel{\alpha=\mathtt{b}}{\longleftarrow} 4\gamma\mathtt{cec\underline{a}} \\ \stackrel{\alpha=\mathtt{c}}{\longleftarrow} 2\gamma\mathtt{cec\underline{e}} \\ \stackrel{\alpha=\mathtt{e}}{\longleftarrow} 5\gamma\mathtt{cec\underline{b}} \end{array} \right. \\ \stackrel{\alpha_1=\mathtt{e},\alpha=\mathtt{c}}{\longleftarrow} & \left\{\ 3\gamma\mathtt{cea\underline{d}} \quad 4\gamma\mathtt{cea\underline{d}}\ \right\} \end{array} \right. \end{array} \right. \tag{36}$$

Continuing in the same way from the next part of (13) note that for the following combinations of γ and α, LR rules result from the LHS: $\gamma \in \{\mathtt{a}, \mathtt{c}\}$ with $\alpha \in \{\mathtt{a}, \mathtt{d}, \mathtt{e}\}$ and for $\gamma = \mathtt{d}$ with $\alpha \in \{\mathtt{a}, \mathtt{b}, \mathtt{c}, \mathtt{d}, \mathtt{e}\}$. The reverse rules needed to complete the derivations of the origins are given at the end. This results in the following LHS's of IRR(5) and their origins corresponding to the second part of (13)

$$2\underline{\gamma}\mathtt{ec}\alpha_1\alpha \left\{ \begin{array}{ll} \stackrel{\alpha_1=\mathtt{a},\alpha=\mathtt{c}}{\longleftarrow} & \left\{ \begin{array}{l} 3\gamma\mathtt{add\underline{d}} \\ 4\gamma\mathtt{add\underline{d}} \end{array} \right. \\ \stackrel{\alpha_1=\mathtt{b}}{\longleftarrow} & \left\{ \begin{array}{l} \stackrel{\alpha=\mathtt{a}}{\longleftarrow} 5\gamma\mathtt{ada\underline{d}} \\ \stackrel{\alpha=\mathtt{c}}{\longleftarrow} \left\{\ 2\gamma\mathtt{ada\underline{b}} \quad 3\gamma\mathtt{ada\underline{b}}\ \right\} \\ \stackrel{\alpha=\mathtt{d}}{\longleftarrow} 1\gamma\mathtt{ada\underline{b}} \end{array} \right. \\ \stackrel{\alpha_1=\mathtt{c}}{\longleftarrow} & \left\{ \begin{array}{l} \stackrel{\alpha=\mathtt{a}}{\longleftarrow} \left\{\ 3\gamma\mathtt{ade\underline{c}} \quad 3\gamma\mathtt{adb\underline{c}} \quad 1\gamma\mathtt{adb\underline{c}}\ \right\} \\ \stackrel{\alpha=\mathtt{b}}{\longleftarrow} 4\gamma\mathtt{adb\underline{a}} \\ \stackrel{\alpha=\mathtt{c}}{\longleftarrow} 2\gamma\mathtt{adb\underline{e}} \\ \stackrel{\alpha=\mathtt{d}}{\longleftarrow} \left\{\ 1\gamma\mathtt{ade\underline{a}} \quad 1\gamma\mathtt{adb\underline{a}}\ \right\} \\ \stackrel{\alpha=\mathtt{e}}{\longleftarrow} \left\{\ 5\gamma\mathtt{ade\underline{a}} \quad 5\gamma\mathtt{adb\underline{a}} \quad 5\gamma\mathtt{adb\underline{b}}\ \right\} \end{array} \right. \\ \stackrel{\alpha_1=\mathtt{e},\alpha=\mathtt{c}}{\longleftarrow} & \left\{\ 3\gamma\mathtt{adb\underline{d}} \quad 4\gamma\mathtt{adb\underline{d}}\ \right\} \end{array} \right. \quad \begin{array}{l} \text{for } \gamma \in \{\mathtt{a}, \mathtt{c}, \mathtt{d}\} \\ \text{and if } \gamma = \mathtt{d} \text{ then} \\ \alpha_1 \notin \{\mathtt{b}, \mathtt{c}\} \end{array} \tag{37}$$

Continuing to the third part of (13) likewise gives

and the LHS's of IRR(5) and their origins corresponding to the final part of (13) are

$$\texttt{2}\underline{\texttt{d}}\texttt{aec}\alpha \begin{cases} \overset{\alpha=\texttt{a}}{\longleftarrow} \left\{ \texttt{1dcbd}\underline{\texttt{c}} \quad \texttt{5dcbd}\underline{\texttt{d}} \right\} \\ \overset{\alpha=\texttt{b}}{\longleftarrow} \texttt{4dcbd}\underline{\texttt{a}} \\ \overset{\alpha=\texttt{c}}{\longleftarrow} \left\{ \texttt{2dcbd}\underline{\texttt{e}} \quad \texttt{2dcbd}\underline{\texttt{b}} \quad \texttt{3dcbd}\underline{\texttt{b}} \right\} \\ \overset{\alpha=\texttt{e}}{\longleftarrow} \left\{ \texttt{5dcbd}\underline{\texttt{b}} \quad \texttt{1dcbd}\underline{\texttt{b}} \right\} \end{cases} \tag{38}$$

Likewise for extending (14) gives

$$\texttt{2}\alpha\texttt{bab}\underline{\texttt{e}} \begin{cases} \overset{\alpha=\texttt{a}}{\longleftarrow} \texttt{2}\underline{\texttt{d}}\gamma_1\texttt{d}\gamma_2\texttt{e} \\ \overset{\alpha=\texttt{c}}{\longleftarrow} \texttt{2}\underline{\texttt{a}}\gamma_1\texttt{d}\gamma_2\texttt{e} \\ \overset{\alpha=\texttt{d}}{\longleftarrow} \texttt{2}\underline{\texttt{c}}\gamma_1\texttt{d}\gamma_2\texttt{e} \end{cases} \tag{39}$$

$$\texttt{2}\alpha\texttt{cab}\underline{\texttt{e}} \begin{cases} \overset{\alpha=\texttt{a}}{\longleftarrow} \left\{ \texttt{5}\underline{\texttt{c}}\texttt{ac}\delta\texttt{e} \quad \texttt{5}\underline{\texttt{e}}\texttt{ac}\delta\texttt{e} \right\} \\ \overset{\alpha=\texttt{b}}{\longleftarrow} \texttt{3}\underline{\texttt{e}}\texttt{ac}\delta\texttt{e} \\ \overset{\alpha=\texttt{c}}{\longleftarrow} \texttt{4}\underline{\texttt{c}}\texttt{ac}\delta\texttt{e} \end{cases} \tag{40}$$

$$\texttt{2}\alpha\texttt{adb}\underline{\texttt{e}} \begin{cases} \overset{\alpha=\texttt{a}}{\longleftarrow} \left\{ \texttt{4}\underline{\texttt{e}}\texttt{cade} \quad \texttt{5}\underline{\texttt{c}}\texttt{eade} \quad \texttt{5}\underline{\texttt{e}}\texttt{eade} \right\} \\ \overset{\alpha=\texttt{b}}{\longleftarrow} \left\{ \texttt{4}\underline{\texttt{b}}\texttt{e}\zeta\texttt{ce} \quad \texttt{3}\underline{\texttt{e}}\texttt{eade} \right\} \\ \overset{\alpha=\texttt{c}}{\longleftarrow} \left\{ \texttt{3}\underline{\texttt{a}}\texttt{e}\zeta\texttt{ce} \quad \texttt{4}\underline{\texttt{c}}\texttt{eade} \right\} \\ \overset{\zeta=\texttt{c}}{\longleftarrow} \texttt{5}\alpha\texttt{ebe}\underline{\texttt{a}} \end{cases} \tag{41}$$

$$\texttt{2}\alpha\texttt{bdb}\underline{\texttt{e}} \begin{cases} \overset{\alpha=\texttt{a}}{\longleftarrow} \left\{ \begin{matrix} \texttt{2}\underline{\texttt{d}}\gamma_1\texttt{c}\gamma_2\texttt{e} & \texttt{2}\underline{\texttt{d}}\gamma_1\texttt{abe} & \texttt{2}\underline{\texttt{d}}\gamma_1\texttt{aae} & \texttt{2}\underline{\texttt{d}}\gamma_3\texttt{bde} \\ \texttt{5}\underline{\texttt{c}}\texttt{ebde} & \texttt{5}\underline{\texttt{e}}\texttt{ebde} & \texttt{5}\underline{\texttt{c}}\texttt{eade} & \texttt{5}\underline{\texttt{e}}\texttt{eade} \end{matrix} \right\} \\ \overset{\alpha=\texttt{b}}{\longleftarrow} \left\{ \texttt{4}\underline{\texttt{b}}\texttt{bcde} \quad \texttt{3}\underline{\texttt{e}}\texttt{ebde} \quad \texttt{3}\underline{\texttt{e}}\texttt{eade} \right\} \\ \overset{\alpha=\texttt{c}}{\longleftarrow} \left\{ \begin{matrix} \texttt{2}\underline{\texttt{a}}\gamma_1\texttt{c}\gamma_2\texttt{e} & \texttt{2}\underline{\texttt{a}}\gamma_1\texttt{abe} & \texttt{2}\underline{\texttt{a}}\gamma_1\texttt{aae} & \texttt{3}\underline{\texttt{a}}\texttt{bcde} \\ \texttt{2}\underline{\texttt{a}}\gamma_3\texttt{bde} & \texttt{4}\underline{\texttt{c}}\texttt{ebde} & \texttt{4}\underline{\texttt{c}}\texttt{eade} & \end{matrix} \right\} \\ \overset{\alpha=\texttt{d}}{\longleftarrow} \left\{ \texttt{2}\underline{\texttt{c}}\gamma_1\texttt{c}\gamma_2\texttt{e} \quad \texttt{2}\underline{\texttt{c}}\gamma_1\texttt{abe} \quad \texttt{2}\underline{\texttt{c}}\gamma_1\texttt{aae} \quad \texttt{2}\underline{\texttt{c}}\gamma_3\texttt{bde} \right\} \end{cases} \tag{42}$$

$$\texttt{2}\alpha\texttt{cdb}\underline{\texttt{e}} \begin{cases} \overset{\alpha=\texttt{a}}{\longleftarrow} \left\{ \texttt{5}\underline{\texttt{c}}\texttt{acde} \quad \texttt{5}\underline{\texttt{e}}\texttt{acde} \right\} \\ \overset{\alpha=\texttt{b}}{\longleftarrow} \left\{ \texttt{3}\underline{\texttt{e}}\texttt{acde} \quad \texttt{4}\underline{\texttt{b}}\texttt{cade} \right\} \\ \overset{\alpha=\texttt{c}}{\longleftarrow} \left\{ \texttt{4}\underline{\texttt{c}}\texttt{acde} \quad \texttt{3}\underline{\texttt{a}}\texttt{cade} \right\} \end{cases} \tag{43}$$

The following is the list of (AIRR) generated during the above calculations.

This starts with the arguments used to generate the LHS's of the IRR(4) from those of the IRR(3) involve use of re-usable reverse rules. These will be listed below, first those of length 3, then length 4. Restrictions on the values of parameters arise from the parameters used in the reverse rules to be derived, so more general statements can sometimes be made, but this is not the point of these results. Sorting them into some sort of order facilitated removal of redundancy and some summarisation.

The list of AIRR obtained:

$$\mathtt{1}\alpha\underline{\gamma}\beta \begin{cases} \overset{\alpha=\mathtt{a}}{\longleftarrow} \mathtt{2}\underline{\mathtt{d}}\gamma\beta \\ \overset{\alpha=\mathtt{c}}{\longleftarrow} \mathtt{2}\underline{\mathtt{a}}\gamma\beta \\ \overset{\alpha=\mathtt{d}}{\longleftarrow} \mathtt{2}\underline{\mathtt{c}}\gamma\beta \end{cases} \tag{44}$$

$$\mathtt{2}\alpha\underline{\gamma}\beta \begin{cases} \overset{\beta=\mathtt{a}}{\longleftarrow} \mathtt{3}\alpha\gamma\underline{\mathtt{c}} \\ \overset{\beta=\mathtt{d}}{\longleftarrow} \mathtt{1}\alpha\gamma\underline{\mathtt{a}} \\ \overset{\beta=\mathtt{e}}{\longleftarrow} \mathtt{5}\alpha\gamma\underline{\mathtt{a}} \\ \overset{\alpha=\mathtt{b}}{\longleftarrow} \left\{ \ \mathtt{1}\underline{\mathtt{d}}\gamma\beta \quad \mathtt{1}\underline{\mathtt{e}}\gamma\beta \ \right\} \end{cases} \quad * \tag{45}$$

$$\mathtt{3}\alpha\underline{\gamma}\beta \begin{cases} \overset{\alpha=\mathtt{a}}{\longleftarrow} \left\{ \ \mathtt{5}\underline{\mathtt{c}}\gamma\beta \quad \mathtt{5}\underline{\gamma}\mathtt{e}\beta \ \right\} \\ \overset{\alpha=\mathtt{b}}{\longleftarrow} \mathtt{3}\underline{\gamma}\mathtt{e}\beta \\ \overset{\alpha=\mathtt{c}}{\longleftarrow} \mathtt{4}\underline{\mathtt{c}}\gamma\beta \\ \overset{\beta=\mathtt{a}}{\longleftarrow} \mathtt{1}\alpha\gamma\underline{\mathtt{c}} \\ \overset{\beta=\mathtt{b}}{\longleftarrow} \mathtt{4}\alpha\gamma\underline{\mathtt{a}} \\ \overset{\beta=\mathtt{c}}{\longleftarrow} \mathtt{2}\alpha\gamma\underline{\mathtt{e}} \\ \overset{\beta=\mathtt{e}}{\longleftarrow} \mathtt{5}\alpha\gamma\underline{\mathtt{b}} \end{cases} \quad * \tag{46}$$

$$\mathtt{4}\alpha\underline{\gamma}\beta \begin{cases} \overset{\alpha=\mathtt{b}}{\longleftarrow} \mathtt{4}\underline{\mathtt{b}}\gamma\beta \\ \overset{\alpha=\mathtt{c}}{\longleftarrow} \mathtt{3}\underline{\mathtt{a}}\gamma\beta \\ \overset{\beta=\mathtt{a}}{\longleftarrow} \mathtt{5}\alpha\gamma\underline{\mathtt{d}} \\ \overset{\beta=\mathtt{c}}{\longleftarrow} \left\{ \ \mathtt{2}\alpha\gamma\underline{\mathtt{b}} \quad \mathtt{3}\alpha\gamma\underline{\mathtt{b}} \ \right\} \\ \overset{\beta=\mathtt{d}}{\longleftarrow} \mathtt{1}\alpha\gamma\underline{\mathtt{b}} \end{cases} \quad * \tag{47}$$

$$\mathtt{5}\alpha\underline{\gamma}\beta \begin{cases} \overset{\alpha=\mathtt{a}}{\longleftarrow} \mathtt{4}\underline{\mathtt{e}}\gamma\beta \\ \overset{\beta=\mathtt{c}}{\longleftarrow} \left\{ \ \mathtt{3}\alpha\gamma\underline{\mathtt{d}} \quad \mathtt{4}\alpha\gamma\underline{\mathtt{d}} \ \right\} \end{cases} \quad * \tag{48}$$

AIRR of length 4

$$\mathtt{1}\gamma\mathtt{a}\underline{\mathtt{b}}\alpha \overset{\gamma=\mathtt{b}}{\longleftarrow} \left\{ \ \mathtt{1}\underline{\mathtt{d}}\mathtt{db}\alpha \quad \mathtt{1}\underline{\mathtt{e}}\mathtt{db}\alpha \ \right\} \tag{49}$$

$$\mathtt{1}\gamma\mathtt{c}\underline{\mathtt{c}}\alpha \overset{\gamma=\mathtt{b}}{\longleftarrow} \left\{ \ \mathtt{1}\underline{\mathtt{d}}\mathtt{ac}\alpha \quad \mathtt{1}\underline{\mathtt{e}}\mathtt{ac}\alpha \ \right\} \tag{50}$$

$$\mathtt{1}\beta\mathtt{d}\underline{\gamma}\alpha \leftarrow \emptyset \text{ for } \beta \in \{\mathtt{a},\mathtt{c},\mathtt{d},\mathtt{e}\} \text{ and } \gamma \in \{\mathtt{b},\mathtt{c}\} \tag{51}$$

$$2\delta\mathtt{b}\underline{\mathtt{e}}\alpha \left\{ \begin{array}{l} \xleftarrow{\alpha=\mathtt{a}} 3\delta\mathtt{be}\underline{\mathtt{c}} \\ \xleftarrow{\alpha=\mathtt{d}} 1\delta\mathtt{be}\underline{\mathtt{a}} \\ \xleftarrow{\alpha=\mathtt{e}} 5\delta\mathtt{be}\underline{\mathtt{a}} \\ \xleftarrow{\delta=\mathtt{a}} \left\{\ 2\underline{\mathtt{d}}\mathtt{de}\alpha \quad 2\underline{\mathtt{d}}\mathtt{ee}\alpha\ \right\} \\ \xleftarrow{\delta=\mathtt{c}} \left\{\ 2\underline{\mathtt{a}}\mathtt{de}\alpha \quad 2\underline{\mathtt{a}}\mathtt{ee}\alpha\ \right\} \\ \xleftarrow{\delta=\mathtt{d}} \left\{\ 2\underline{\mathtt{c}}\mathtt{de}\alpha \quad 2\underline{\mathtt{c}}\mathtt{ee}\alpha\ \right\} \end{array}\right. \qquad * \tag{52}$$

$$3\alpha\underline{\mathtt{a}}\mathtt{ba} \left\{ \begin{array}{l} \xleftarrow{\alpha=\mathtt{a}} \left\{\ 5\underline{\mathtt{c}}\mathtt{aba} \quad 5\underline{\mathtt{e}}\mathtt{aba}\ \right\} \\ \xleftarrow{\alpha=\mathtt{b}} 3\underline{\mathtt{e}}\mathtt{aba} \\ \xleftarrow{\alpha=\mathtt{c}} 4\underline{\mathtt{c}}\mathtt{aba} \\ \leftarrow 5\alpha\mathtt{aa}\underline{\mathtt{d}} \end{array}\right. \qquad * \tag{53}$$

$$3\alpha\underline{\mathtt{a}}\mathtt{bb} \left\{ \begin{array}{l} \xleftarrow{\alpha=\mathtt{a}} \left\{\ 5\underline{\mathtt{c}}\mathtt{abb} \quad 5\underline{\mathtt{e}}\mathtt{abb}\ \right\} \\ \xleftarrow{\alpha=\mathtt{b}} 3\underline{\mathtt{e}}\mathtt{abb} \\ \xleftarrow{\alpha=\mathtt{c}} 4\underline{\mathtt{c}}\mathtt{abb} \end{array}\right. \tag{54}$$

$$3\alpha\underline{\mathtt{a}}\mathtt{bd} \left\{ \begin{array}{l} 1\alpha\mathtt{aa}\underline{\mathtt{b}} \\ \xleftarrow{\alpha=\mathtt{a}} \left\{ \begin{array}{l} 5\underline{\mathtt{c}}\mathtt{abd} \\ 5\underline{\mathtt{e}}\mathtt{abd} \end{array}\right. \\ \xleftarrow{\alpha=\mathtt{b}} 3\underline{\mathtt{e}}\mathtt{abd} \\ \xleftarrow{\alpha=\mathtt{c}} 4\underline{\mathtt{c}}\mathtt{abd} \end{array}\right. \qquad * \tag{55}$$

$$3\alpha\underline{\mathtt{a}}\mathtt{be} \left\{ \begin{array}{l} \xleftarrow{\alpha=\mathtt{a}} \left\{ \begin{array}{l} 5\underline{\mathtt{c}}\mathtt{abe} \\ 5\underline{\mathtt{e}}\mathtt{abe} \end{array}\right. \\ \xleftarrow{\alpha=\mathtt{b}} 3\underline{\mathtt{e}}\mathtt{abe} \\ \xleftarrow{\alpha=\mathtt{c}} 4\underline{\mathtt{c}}\mathtt{abe} \end{array}\right. \tag{56}$$

$$3\alpha\underline{\mathtt{a}}\mathtt{c}\delta \left\{ \begin{array}{l} \xleftarrow{\alpha=\mathtt{a}} \left\{\ 5\underline{\mathtt{c}}\mathtt{ac}\delta \quad 5\underline{\mathtt{e}}\mathtt{ac}\delta\ \right\} \\ \xleftarrow{\alpha=\mathtt{b}} 3\underline{\mathtt{e}}\mathtt{ac}\delta \\ \xleftarrow{\alpha=\mathtt{c}} 4\underline{\mathtt{c}}\mathtt{ac}\delta \end{array}\right. \qquad \text{for } \delta \in \{\mathtt{b}, \mathtt{c}\} \tag{57}$$

$$3\alpha\underline{\mathtt{a}}\mathtt{e}\beta \left\{ \begin{array}{l} \xleftarrow{\alpha=\mathtt{a}} \left\{ \begin{array}{l} 5\underline{\mathtt{c}}\mathtt{ae}\beta \\ 5\underline{\mathtt{e}}\mathtt{ae}\beta \end{array}\right. \\ \xleftarrow{\alpha=\mathtt{b}} \left\{ \begin{array}{l} 3\underline{\mathtt{e}}\mathtt{ae}\beta \\ 4\underline{\mathtt{b}}\mathtt{eb}\beta \end{array}\right. \\ \xleftarrow{\alpha=\mathtt{c}} \left\{ \begin{array}{l} 4\underline{\mathtt{c}}\mathtt{ae}\beta \\ 3\underline{\mathtt{a}}\mathtt{eb}\beta \end{array}\right. \end{array}\right. \qquad \text{for } \beta \in \{\mathtt{b}, \mathtt{d}\} \tag{58}$$

$$3\alpha\underline{\mathtt{e}}\mathtt{a}\beta \left\{ \begin{array}{l} \xleftarrow{\alpha=\mathtt{a}} \left\{\ 5\underline{\mathtt{c}}\mathtt{ea}\beta \quad 5\underline{\mathtt{e}}\mathtt{ea}\beta\ \right\} \\ \xleftarrow{\alpha=\mathtt{b}} 3\underline{\mathtt{e}}\mathtt{ea}\beta \\ \xleftarrow{\alpha=\mathtt{c}} 4\underline{\mathtt{c}}\mathtt{ea}\beta \end{array}\right. \tag{59}$$

$$\mathtt{3}\alpha\underline{\mathtt{e}}\mathtt{e}\gamma \begin{cases} \overset{\alpha=\mathtt{a}}{\longleftarrow} \left\{ \mathtt{5}\underline{\mathtt{c}}\mathtt{ee}\gamma \quad \mathtt{5}\underline{\mathtt{e}}\mathtt{ee}\gamma \right\} \\ \overset{\alpha=\mathtt{b}}{\longleftarrow} \mathtt{3}\underline{\mathtt{e}}\mathtt{ee}\gamma \\ \overset{\alpha=\mathtt{c}}{\longleftarrow} \mathtt{4}\underline{\mathtt{c}}\mathtt{ee}\gamma \\ \overset{\gamma=\mathtt{c}}{\longleftarrow} \left\{ \mathtt{3}\alpha\mathtt{eb}\underline{\mathtt{d}} \quad \mathtt{4}\alpha\mathtt{eb}\underline{\mathtt{d}} \right\} \end{cases} \qquad \text{for } \gamma \in \{\mathtt{c},\mathtt{d},\mathtt{e}\} \ * \tag{60}$$

$$\mathtt{3}\alpha\underline{\mathtt{e}}\mathtt{a}\theta \begin{cases} \overset{\alpha=\mathtt{a}}{\longleftarrow} \begin{cases} \mathtt{5}\underline{\mathtt{c}}\mathtt{ea}\theta \\ \mathtt{5}\underline{\mathtt{e}}\mathtt{ea}\theta \end{cases} \\ \overset{\alpha=\mathtt{b}}{\longleftarrow} \mathtt{3}\underline{\mathtt{e}}\mathtt{ea}\theta \\ \overset{\alpha=\mathtt{c}}{\longleftarrow} \mathtt{4}\underline{\mathtt{c}}\mathtt{ea}\theta \end{cases} \qquad \text{for } \theta \in \{\mathtt{a},\mathtt{b},\mathtt{d}\} \tag{61}$$

$$\mathtt{3da}\underline{\mathtt{b}}\alpha \begin{cases} \overset{\alpha=\mathtt{a}}{\longleftarrow} \left\{ \mathtt{1dab}\underline{\mathtt{c}} \quad \mathtt{5dca}\underline{\mathtt{d}} \right\} \\ \overset{\alpha=\mathtt{b}}{\longleftarrow} \mathtt{4dab}\underline{\mathtt{a}} \\ \overset{\alpha=\mathtt{c}}{\longleftarrow} \left\{ \mathtt{2dab}\underline{\mathtt{e}} \quad \mathtt{2dca}\underline{\mathtt{b}} \quad \mathtt{3dca}\underline{\mathtt{b}} \right\} \\ \overset{\alpha=\mathtt{d}}{\longleftarrow} \mathtt{1cda}\underline{\mathtt{b}} \\ \overset{\alpha=\mathtt{e}}{\longleftarrow} \mathtt{5dab}\underline{\mathtt{b}} \end{cases} \tag{62}$$

$$\mathtt{3}\gamma\mathtt{a}\underline{\mathtt{b}}\alpha \begin{cases} \overset{\alpha=\mathtt{a}}{\longleftarrow} \mathtt{1}\gamma\mathtt{ab}\underline{\mathtt{c}} \\ \overset{\alpha=\mathtt{b}}{\longleftarrow} \mathtt{4}\gamma\mathtt{ab}\underline{\mathtt{a}} \\ \overset{\alpha=\mathtt{c}}{\longleftarrow} \mathtt{2}\gamma\mathtt{ab}\underline{\mathtt{e}} \\ \overset{\alpha=\mathtt{e}}{\longleftarrow} \mathtt{5}\gamma\mathtt{ab}\underline{\mathtt{b}} \end{cases} \qquad \text{for } \gamma \in \{\mathtt{a},\mathtt{b},\mathtt{c},\mathtt{e}\} \tag{63}$$

$$\mathtt{3d}\epsilon\underline{\mathtt{c}}\alpha \begin{cases} \overset{\alpha=\mathtt{a}}{\longleftarrow} \mathtt{1d}\epsilon\mathtt{c}\underline{\mathtt{c}} \\ \overset{\alpha=\mathtt{b}}{\longleftarrow} \mathtt{4d}\epsilon\mathtt{c}\underline{\mathtt{a}} \\ \overset{\alpha=\mathtt{c}}{\longleftarrow} \mathtt{2d}\epsilon\mathtt{c}\underline{\mathtt{e}} \\ \overset{\alpha=\mathtt{e}}{\longleftarrow} \mathtt{5d}\epsilon\mathtt{c}\underline{\mathtt{b}} \\ \overset{\epsilon=\mathtt{b}}{\longleftarrow} \begin{cases} \overset{\alpha=\mathtt{a}}{\longleftarrow} \mathtt{3dee}\underline{\mathtt{c}} \\ \overset{\alpha=\mathtt{d}}{\longleftarrow} \mathtt{1dee}\underline{\mathtt{a}} \\ \overset{\alpha=\mathtt{e}}{\longleftarrow} \mathtt{5dee}\underline{\mathtt{a}} \end{cases} \end{cases} \qquad \text{for } \epsilon \in \{\mathtt{b},\mathtt{e}\} \tag{64}$$

$$\mathtt{3}\delta\mathtt{b}\underline{\mathtt{c}}\alpha \begin{cases} \overset{\alpha=\mathtt{a}}{\longleftarrow} \begin{cases} \mathtt{1}\delta\mathtt{bc}\underline{\mathtt{c}} \\ \mathtt{3}\delta\mathtt{ee}\underline{\mathtt{c}} \end{cases} \\ \overset{\alpha=\mathtt{b}}{\longleftarrow} \mathtt{4}\delta\mathtt{bc}\underline{\mathtt{a}} \\ \overset{\alpha=\mathtt{c}}{\longleftarrow} \mathtt{2}\delta\mathtt{bc}\underline{\mathtt{e}} \\ \overset{\alpha=\mathtt{d}}{\longleftarrow} \mathtt{1}\delta\mathtt{ee}\underline{\mathtt{a}} \\ \overset{\alpha=\mathtt{e}}{\longleftarrow} \begin{cases} \mathtt{5}\delta\mathtt{bc}\underline{\mathtt{b}} \\ \mathtt{5}\delta\mathtt{ee}\underline{\mathtt{a}} \end{cases} \end{cases} \qquad \text{for } \delta \in \{\mathtt{b},\mathtt{c},\mathtt{e}\} \tag{65}$$

$$\mathtt{3}\beta\mathtt{a\underline{d}}\alpha \begin{cases} \overset{\alpha=\mathtt{a}}{\longleftarrow} \mathtt{1}\beta\mathtt{ad\underline{c}} \\ \overset{\alpha=\mathtt{b}}{\longleftarrow} \mathtt{4}\beta\mathtt{ad\underline{a}} \\ \overset{\alpha=\mathtt{c}}{\longleftarrow} \mathtt{2}\beta\mathtt{ad\underline{e}} \\ \overset{\alpha=\mathtt{e}}{\longleftarrow} \mathtt{5}\beta\mathtt{ad\underline{b}} \\ \overset{\beta=\mathtt{a}}{\longleftarrow} \left\{ \mathtt{4\underline{e}cd}\alpha \quad \mathtt{4\underline{e}ed}\alpha \right\} \end{cases} \qquad * \tag{66}$$

$$\mathtt{3}\gamma\mathtt{b\underline{d}}\alpha \begin{cases} \overset{\alpha=\mathtt{a}}{\longleftarrow} \mathtt{1}\gamma\mathtt{bd\underline{c}} \\ \overset{\alpha=\mathtt{b}}{\longleftarrow} \mathtt{4}\gamma\mathtt{bd\underline{a}} \\ \overset{\alpha=\mathtt{c}}{\longleftarrow} \mathtt{2}\gamma\mathtt{bd\underline{e}} \\ \overset{\alpha=\mathtt{e}}{\longleftarrow} \mathtt{5}\gamma\mathtt{bd\underline{b}} \end{cases} \quad \text{for } \gamma \in \{\mathtt{a}, \mathtt{e}\} \tag{67}$$

$$\mathtt{4}\alpha\mathtt{\underline{b}cb} \begin{cases} \leftarrow \mathtt{4}\alpha\mathtt{bb\underline{a}}\ 5{*}(90) \\ \overset{\alpha=\mathtt{a}}{\longleftarrow} \left\{ \mathtt{2\underline{d}dbb} \quad \mathtt{2\underline{d}ebb} \quad \mathtt{5\underline{c}ebb} \quad \mathtt{5\underline{e}ebb} \right\} \\ \overset{\alpha=\mathtt{b}}{\longleftarrow} \left\{ \mathtt{4\underline{b}bcb} \quad \mathtt{3\underline{e}ebb} \right\} \\ \overset{\alpha=\mathtt{c}}{\longleftarrow} \left\{ \mathtt{3\underline{a}bcb} \quad \mathtt{2\underline{a}dbb} \quad \mathtt{2\underline{a}ebb} \quad \mathtt{4\underline{c}ebb} \right\} \\ \overset{\alpha=\mathtt{d}}{\longleftarrow} \left\{ \mathtt{2\underline{c}dbb} \quad \mathtt{2\underline{c}ebb} \right\} \end{cases} \qquad * \tag{68}$$

$$\mathtt{4}\alpha\mathtt{\underline{b}cc} \begin{cases} \leftarrow \left\{ \mathtt{2}\alpha\mathtt{bb\underline{e}}\ 4(52) \quad \mathtt{2}\alpha\mathtt{ea\underline{b}}\ 3(45) \quad \mathtt{3}\alpha\mathtt{ea\underline{b}}\ 4(63) \right\} \\ \overset{\alpha=\mathtt{a}}{\longleftarrow} \left\{ \mathtt{2\underline{d}dbc} \quad \mathtt{2\underline{d}ebc} \quad \mathtt{5\underline{c}ebc} \quad \mathtt{5\underline{e}ebc} \right\} \\ \overset{\alpha=\mathtt{b}}{\longleftarrow} \left\{ \mathtt{3\underline{e}ebc} \quad \mathtt{4\underline{b}bcc} \right\} \\ \overset{\alpha=\mathtt{c}}{\longleftarrow} \left\{ \mathtt{2\underline{a}dbc} \quad \mathtt{2\underline{a}ebc} \quad \mathtt{4\underline{c}ebc} \quad \mathtt{3\underline{a}bcc} \right\} \\ \overset{\alpha=\mathtt{d}}{\longleftarrow} \left\{ \mathtt{2\underline{c}dbc} \quad \mathtt{2\underline{c}ebc} \right\} \end{cases} \qquad * \tag{69}$$

$$\mathtt{4}\alpha\mathtt{\underline{c}a}\beta \begin{cases} \overset{\alpha=\mathtt{b}}{\longleftarrow} \mathtt{4\underline{b}ca}\beta \\ \overset{\alpha=\mathtt{c}}{\longleftarrow} \mathtt{3\underline{a}ca}\beta \end{cases} \quad \text{for } \beta \in \{\mathtt{a}, \mathtt{b}, \mathtt{d}, \mathtt{e}\} \tag{70}$$

$$\mathtt{4}\alpha\mathtt{\underline{e}c}\gamma \begin{cases} \overset{\alpha=\mathtt{b}}{\longleftarrow} \mathtt{4\underline{b}ec}\gamma \\ \overset{\alpha=\mathtt{c}}{\longleftarrow} \mathtt{3\underline{a}ec}\gamma \\ \overset{\gamma=\mathtt{a}}{\longleftarrow} \left\{ \mathtt{3}\alpha\mathtt{eb\underline{c}} \quad \mathtt{1}\alpha\mathtt{eb\underline{c}} \right\} \\ \overset{\gamma=\mathtt{b}}{\longleftarrow} \mathtt{4}\alpha\mathtt{eb\underline{a}} \\ \overset{\gamma=\mathtt{c}}{\longleftarrow} \mathtt{2}\alpha\mathtt{eb\underline{e}} \\ \overset{\gamma=\mathtt{d}}{\longleftarrow} \mathtt{1}\alpha\mathtt{eb\underline{a}} \\ \overset{\gamma=\mathtt{e}}{\longleftarrow} \left\{ \mathtt{5}\alpha\mathtt{eb\underline{a}} \quad \mathtt{5}\alpha\mathtt{eb\underline{b}} \right\} \end{cases} \qquad * \tag{71}$$

$$\mathtt{4}\beta\mathtt{b\underline{a}}\alpha \begin{cases} \overset{\alpha=\mathtt{a}}{\longleftarrow} \mathtt{5}\beta\mathtt{ba\underline{d}} \\ \overset{\alpha=\mathtt{c}}{\longleftarrow} \begin{cases} \mathtt{2}\beta\mathtt{ba\underline{b}} \\ \mathtt{3}\beta\mathtt{ba\underline{b}} \\ \mathtt{3}\beta\mathtt{bd\underline{d}} \\ \mathtt{4}\beta\mathtt{bd\underline{d}} \end{cases} \\ \overset{\alpha=\mathtt{d}}{\longleftarrow} \mathtt{1}\beta\mathtt{ba\underline{b}} \end{cases} \tag{72}$$

$$\mathtt{4}\gamma\mathtt{c\underline{a}}\alpha \begin{cases} \overset{\alpha=\mathtt{a}}{\leftarrow} \mathtt{5}\gamma\mathtt{ca\underline{d}} \\ \overset{\alpha=\mathtt{c}}{\leftarrow} \begin{cases} \mathtt{2}\gamma\mathtt{ca\underline{b}} \\ \mathtt{3}\gamma\mathtt{ca\underline{b}} \end{cases} \\ \overset{\alpha=\mathtt{d}}{\leftarrow} \mathtt{1}\gamma\mathtt{ca\underline{b}} \\ \overset{\gamma=\mathtt{a}}{\leftarrow} \left\{ \mathtt{5\underline{c}aa}\alpha \quad \mathtt{5\underline{e}aa}\alpha \right\} \\ \overset{\gamma=\mathtt{b}}{\leftarrow} \left\{ \mathtt{3\underline{e}aa}\alpha \quad \mathtt{1\underline{d}dc}\alpha \quad \mathtt{1\underline{e}dc}\alpha \right\} \\ \overset{\gamma=\mathtt{c}}{\leftarrow} \mathtt{4\underline{c}aa}\alpha \end{cases} \tag{73}$$

$$\mathtt{4}\gamma\mathtt{b\underline{d}}\alpha \begin{cases} \overset{\alpha=\mathtt{a}}{\leftarrow} \mathtt{5}\gamma\mathtt{bd\underline{d}} \\ \overset{\alpha=\mathtt{c}}{\leftarrow} \begin{cases} \mathtt{2}\gamma\mathtt{bd\underline{b}} \\ \mathtt{3}\gamma\mathtt{bd\underline{b}} \end{cases} \\ \overset{\alpha=\mathtt{d}}{\leftarrow} \mathtt{1}\gamma\mathtt{bd\underline{b}} \end{cases} \quad \text{for } \gamma \in \{\mathtt{a}, \mathtt{e}\} \tag{74}$$

$$\mathtt{5}\gamma\mathtt{a\underline{d}}\alpha \begin{cases} \overset{\alpha=\mathtt{c}}{\leftarrow} \begin{cases} \mathtt{3}\gamma\mathtt{ad\underline{d}} \\ \mathtt{5}\gamma\mathtt{ad\underline{d}} \end{cases} \\ \overset{\gamma=\mathtt{b}}{\leftarrow} \mathtt{4\underline{b}ed}\alpha \\ \overset{\gamma=\mathtt{c}}{\leftarrow} \mathtt{3\underline{a}ed}\alpha \end{cases} \tag{75}$$

AIRR of length 5

$$\mathtt{1}\delta\mathtt{ba\underline{b}}\alpha \overset{\delta=\mathtt{c}}{\leftarrow} \left\{ \mathtt{2\underline{a}ddb}\alpha \quad \mathtt{2\underline{a}edb}\alpha \right\} * \text{ for } \delta \in \{\mathtt{b}, \mathtt{c}, \mathtt{e}\} \tag{76}$$

$$\mathtt{1abd\underline{b}}\alpha \leftarrow \left\{ \mathtt{2\underline{d}dcb}\alpha \quad \mathtt{2\underline{d}ecb}\alpha \right\} * \tag{77}$$

$$\mathtt{1}\delta\gamma\mathtt{c\underline{c}}\alpha \overset{\gamma=\mathtt{b}}{\leftarrow} \begin{cases} \overset{\delta=\mathtt{a}}{\leftarrow} \left\{ \mathtt{2\underline{d}dac}\alpha \quad \mathtt{2\underline{d}eac}\alpha \right\} \\ \overset{\delta=\mathtt{c}}{\leftarrow} \left\{ \mathtt{2\underline{a}dac}\alpha \quad \mathtt{2\underline{a}eac}\alpha \right\} \\ \overset{\delta=\mathtt{d}}{\leftarrow} \left\{ \mathtt{2\underline{c}dac}\alpha \quad \mathtt{2\underline{c}eac}\alpha \right\} \end{cases} \tag{78}$$

$$\mathtt{1ebd\underline{b}}\alpha \leftarrow \emptyset \tag{79}$$

$$\mathtt{1cbc\underline{c}}\alpha \left\{ \mathtt{2\underline{a}dac}\alpha \quad \mathtt{2\underline{a}eac}\alpha \right\} \tag{80}$$

$$\mathtt{1abd\underline{c}}\alpha \leftarrow \left\{ \mathtt{2\underline{d}dcc}\alpha \quad \mathtt{2\underline{d}ecc}\alpha \right\} * \tag{81}$$

$$\mathtt{1ebd\underline{c}}\alpha \leftarrow \emptyset \tag{82}$$

$$2\gamma\mathtt{ab}\underline{\mathtt{e}}\alpha \begin{cases} \overset{\alpha=\mathtt{a}}{\longleftarrow} \left\{\ 3\gamma\mathtt{abe}\underline{\mathtt{c}} \quad 4\gamma\mathtt{cba}\underline{\mathtt{e}}\ \right\} \\ \overset{\alpha=\mathtt{c}}{\longleftarrow} \left\{\ 3\gamma\mathtt{ccb}\underline{\mathtt{d}} \quad 4\gamma\mathtt{ccb}\underline{\mathtt{d}} \quad 3\gamma\mathtt{cbb}\underline{\mathtt{d}} \quad 4\gamma\mathtt{cbb}\underline{\mathtt{d}}\ \right\} \\ \overset{\alpha=\mathtt{d}}{\longleftarrow} 1\gamma\mathtt{abe}\underline{\mathtt{a}} \\ \overset{\alpha=\mathtt{e}}{\longleftarrow} 5\gamma\mathtt{abe}\underline{\mathtt{a}} \\ \overset{\gamma=\mathtt{b}}{\longleftarrow} \left\{ \begin{matrix} 1\underline{\mathtt{d}}\mathtt{dde}\alpha & 1\underline{\mathtt{e}}\mathtt{dde}\alpha & 1\underline{\mathtt{d}}\mathtt{cae}\alpha & 1\underline{\mathtt{e}}\mathtt{cae}\alpha \\ 1\underline{\mathtt{d}}\mathtt{dee}\alpha & 1\underline{\mathtt{e}}\mathtt{dee}\alpha & 4\underline{\mathtt{b}}\mathtt{cce}\alpha & 4\underline{\mathtt{b}}\mathtt{cbe}\alpha \end{matrix} \right\} \\ \overset{\gamma=\mathtt{c}}{\longleftarrow} \left\{\ 3\underline{\mathtt{a}}\mathtt{cce}\alpha \quad 3\underline{\mathtt{a}}\mathtt{cbe}\alpha\ \right\} \end{cases} \qquad * \tag{83}$$

$$2\beta\mathtt{db}\underline{\mathtt{e}}\alpha \begin{cases} \overset{\alpha=\mathtt{a}}{\longleftarrow} 3\beta\mathtt{dbe}\underline{\mathtt{c}} \\ \overset{\alpha=\mathtt{c}}{\longleftarrow} \left\{ \begin{matrix} 3\beta\mathtt{acb}\underline{\mathtt{d}} & 4\beta\mathtt{acb}\underline{\mathtt{d}} & 3\beta\mathtt{cdb}\underline{\mathtt{d}} \\ 4\beta\mathtt{cdb}\underline{\mathtt{d}} & 3\beta\mathtt{edb}\underline{\mathtt{d}} & 4\beta\mathtt{edb}\underline{\mathtt{d}} \end{matrix} \right\} \\ \overset{\alpha=\mathtt{d}}{\longleftarrow} 1\beta\mathtt{dbe}\underline{\mathtt{a}} \\ \overset{\alpha=\mathtt{e}}{\longleftarrow} 5\beta\mathtt{dbe}\underline{\mathtt{a}} \\ \overset{\beta=\mathtt{c}}{\longleftarrow} \left\{\ 3\underline{\mathtt{a}}\mathtt{cde}\alpha \quad 4\underline{\mathtt{c}}\mathtt{ade}\alpha\ \right\} \end{cases} \quad \text{for } \beta \in \{\mathtt{c}, \mathtt{e}\} \tag{84}$$

$$3\alpha\underline{\mathtt{a}}\mathtt{cde} \begin{cases} \overset{\alpha=\mathtt{a}}{\longleftarrow} \left\{\ 5\underline{\mathtt{c}}\mathtt{acde} \quad 5\underline{\mathtt{e}}\mathtt{acde}\ \right\} \\ \overset{\alpha=\mathtt{b}}{\longleftarrow} 3\underline{\mathtt{e}}\mathtt{acde} \\ \overset{\alpha=\mathtt{c}}{\longleftarrow} 4\underline{\mathtt{c}}\mathtt{acde} \end{cases} \tag{85}$$

$$3\alpha\underline{\mathtt{a}}\mathtt{ced} \begin{cases} \overset{\alpha=\mathtt{a}}{\longleftarrow} \left\{\ 5\underline{\mathtt{c}}\mathtt{aced} \quad 5\underline{\mathtt{e}}\mathtt{aced}\ \right\} \\ \overset{\alpha=\mathtt{b}}{\longleftarrow} 3\underline{\mathtt{e}}\mathtt{aced} \\ \overset{\alpha=\mathtt{c}}{\longleftarrow} 4\underline{\mathtt{c}}\mathtt{aced} \end{cases} \tag{86}$$

$$3\gamma\mathtt{ab}\underline{\mathtt{c}}\alpha \begin{cases} \overset{\alpha=\mathtt{a}}{\longleftarrow} \left\{\ 3\gamma\mathtt{aee}\underline{\mathtt{c}} \quad 1\gamma\mathtt{abc}\underline{\mathtt{c}}\ \right\} \\ \overset{\alpha=\mathtt{b}}{\longleftarrow} 4\gamma\mathtt{abc}\underline{\mathtt{a}} \\ \overset{\alpha=\mathtt{c}}{\longleftarrow} 2\gamma\mathtt{abc}\underline{\mathtt{e}} \\ \overset{\alpha=\mathtt{d}}{\longleftarrow} 1\gamma\mathtt{aee}\underline{\mathtt{a}} \\ \overset{\alpha=\mathtt{e}}{\longleftarrow} \left\{\ 5\gamma\mathtt{aee}\underline{\mathtt{a}} \quad 5\gamma\mathtt{abc}\underline{\mathtt{b}}\ \right\} \\ \overset{\gamma=\mathtt{a}}{\longleftarrow} \left\{\ 4\underline{\mathtt{e}}\mathtt{cec}\alpha \quad 4\underline{\mathtt{e}}\mathtt{eec}\alpha\ \right\} \end{cases} \tag{87}$$

$$3\delta\mathtt{bb}\underline{\mathtt{d}}\alpha \begin{cases} \overset{\alpha=\mathtt{a}}{\longleftarrow} 1\delta\mathtt{bbd}\underline{\mathtt{c}} \\ \overset{\alpha=\mathtt{b}}{\longleftarrow} 4\delta\mathtt{bbd}\underline{\mathtt{a}} \\ \overset{\alpha=\mathtt{c}}{\longleftarrow} 2\delta\mathtt{bbd}\underline{\mathtt{e}} \quad * \text{ for } \delta \in \{\mathtt{b}, \mathtt{e}\} \\ \overset{\alpha=\mathtt{e}}{\longleftarrow} 5\delta\mathtt{bbd}\underline{\mathtt{b}} \\ \overset{\delta=\mathtt{b}}{\longleftarrow} 3\underline{\mathtt{e}}\mathtt{eed}\alpha \end{cases} \tag{88}$$

$$3\mathtt{dcb}\underline{\mathtt{d}}\alpha \leftarrow \begin{cases} \overset{\alpha=\mathtt{a}}{\longleftarrow} 1\mathtt{dcbd}\underline{\mathtt{c}} \\ \overset{\alpha=\mathtt{b}}{\longleftarrow} 4\mathtt{dcbd}\underline{\mathtt{a}} \\ \overset{\alpha=\mathtt{c}}{\longleftarrow} 2\mathtt{dcbd}\underline{\mathtt{e}} \\ \overset{\alpha=\mathtt{e}}{\longleftarrow} 5\mathtt{dcbd}\underline{\mathtt{b}} \end{cases} \tag{89}$$

$$
\mathtt{4}\alpha\mathtt{bb}\underline{\mathtt{a}}\epsilon
\begin{cases}
\stackrel{\epsilon=\mathtt{a}}{\longleftarrow} \mathtt{5}\alpha\mathtt{bba}\underline{\mathtt{d}} \\
\stackrel{\epsilon=\mathtt{c}}{\longleftarrow} \left\{\ \mathtt{2}\alpha\mathtt{bba}\underline{\mathtt{b}} \quad \mathtt{3}\alpha\mathtt{bba}\underline{\mathtt{b}}\ \right\} \\
\stackrel{\epsilon=\mathtt{d}}{\longleftarrow} \mathtt{1}\alpha\mathtt{bba}\underline{\mathtt{b}} \\
\stackrel{\alpha=\mathtt{b}}{\longleftarrow} \mathtt{4}\underline{\mathtt{b}}\mathtt{bbae} \\
\stackrel{\alpha=\mathtt{c}}{\longleftarrow} \mathtt{3}\underline{\mathtt{a}}\mathtt{bbae}
\end{cases}
\qquad * \tag{90}
$$

$$
\mathtt{4}\alpha\underline{\mathtt{b}}\mathtt{cc}\epsilon
\begin{cases}
\leftarrow \begin{cases}
\stackrel{\epsilon=\mathtt{a}}{\longleftarrow} \mathtt{3}\alpha\mathtt{bbe}\underline{\mathtt{c}} \\
\stackrel{\epsilon=\mathtt{d}}{\longleftarrow} \mathtt{1}\alpha\mathtt{bbe}\underline{\mathtt{a}} \\
\stackrel{\epsilon=\mathtt{e}}{\longleftarrow} \mathtt{5}\alpha\mathtt{bbe}\underline{\mathtt{a}}
\end{cases} \\
\begin{cases}
\stackrel{\epsilon=\mathtt{a}}{\longleftarrow} \mathtt{3}\alpha\mathtt{eab}\underline{\mathtt{c}} \\
\stackrel{\epsilon=\mathtt{d}}{\longleftarrow} \mathtt{1}\alpha\mathtt{eab}\underline{\mathtt{a}} \\
\stackrel{\epsilon=\mathtt{e}}{\longleftarrow} \mathtt{5}\alpha\mathtt{eab}\underline{\mathtt{a}}
\end{cases} \\
\begin{cases}
\stackrel{\epsilon=\mathtt{a}}{\longleftarrow} \mathtt{1}\alpha\mathtt{eab}\underline{\mathtt{c}} \\
\stackrel{\epsilon=\mathtt{b}}{\longleftarrow} \mathtt{4}\alpha\mathtt{eab}\underline{\mathtt{a}} \\
\stackrel{\epsilon=\mathtt{c}}{\longleftarrow} \mathtt{2}\alpha\mathtt{eab}\underline{\mathtt{e}} \\
\stackrel{\epsilon=\mathtt{e}}{\longleftarrow} \mathtt{5}\alpha\mathtt{eab}\underline{\mathtt{b}}
\end{cases} \\
\stackrel{\alpha=\mathtt{a}}{\longleftarrow} \left\{\ \mathtt{2}\underline{\mathtt{d}}\mathtt{dbc}\epsilon \quad \mathtt{2}\underline{\mathtt{d}}\mathtt{ebc}\epsilon \quad \mathtt{5}\underline{\mathtt{c}}\mathtt{ebc}\epsilon \quad \mathtt{5}\underline{\mathtt{e}}\mathtt{ebc}\epsilon\ \right\} \\
\stackrel{\alpha=\mathtt{b}}{\longleftarrow} \left\{\ \mathtt{3}\underline{\mathtt{e}}\mathtt{ebc}\epsilon \quad \mathtt{4}\underline{\mathtt{b}}\mathtt{bcc}\epsilon\ \right\} \\
\stackrel{\alpha=\mathtt{c}}{\longleftarrow} \left\{\ \mathtt{2}\underline{\mathtt{a}}\mathtt{dbc}\epsilon \quad \mathtt{2}\underline{\mathtt{a}}\mathtt{ebc}\epsilon \quad \mathtt{4}\underline{\mathtt{c}}\mathtt{ebc}\epsilon \quad \mathtt{3}\underline{\mathtt{a}}\mathtt{bcc}\epsilon\ \right\} \\
\stackrel{\alpha=\mathtt{d}}{\longleftarrow} \left\{\ \mathtt{2}\underline{\mathtt{c}}\mathtt{dbc}\epsilon \quad \mathtt{2}\underline{\mathtt{c}}\mathtt{ebc}\epsilon\ \right\}
\end{cases}
\qquad \text{see (69)} \tag{91}
$$

$$
\mathtt{4}\alpha\underline{\mathtt{b}}\mathtt{cd}\epsilon
\begin{cases}
\stackrel{\alpha=\mathtt{a}}{\longleftarrow} \left\{\ \mathtt{2}\underline{\mathtt{d}}\mathtt{dbd}\epsilon \quad \mathtt{2}\underline{\mathtt{d}}\mathtt{ebd}\epsilon \quad \mathtt{5}\underline{\mathtt{c}}\mathtt{ebd}\epsilon \quad \mathtt{5}\underline{\mathtt{e}}\mathtt{ebd}\epsilon\ \right\} \\
\stackrel{\alpha=\mathtt{b}}{\longleftarrow} \left\{\ \mathtt{4}\underline{\mathtt{b}}\mathtt{bcd}\epsilon \quad \mathtt{3}\underline{\mathtt{e}}\mathtt{ebd}\epsilon\ \right\} \\
\stackrel{\alpha=\mathtt{c}}{\longleftarrow} \left\{\ \mathtt{3}\underline{\mathtt{a}}\mathtt{bcd}\epsilon \quad \mathtt{2}\underline{\mathtt{a}}\mathtt{dbd}\epsilon \quad \mathtt{2}\underline{\mathtt{a}}\mathtt{ebd}\epsilon \quad \mathtt{4}\underline{\mathtt{c}}\mathtt{ebd}\epsilon\ \right\} \\
\stackrel{\alpha=\mathtt{d}}{\longleftarrow} \left\{\ \mathtt{2}\underline{\mathtt{c}}\mathtt{dbd}\epsilon \quad \mathtt{2}\underline{\mathtt{c}}\mathtt{ebd}\epsilon\ \right\}
\end{cases}
\tag{92}
$$

$$
\mathtt{4}\alpha\underline{\mathtt{b}}\mathtt{ced}
\begin{cases}
\stackrel{\alpha=\mathtt{a}}{\longleftarrow} \left\{\ \mathtt{2}\underline{\mathtt{d}}\mathtt{dbed} \quad \mathtt{2}\underline{\mathtt{d}}\mathtt{ebed} \quad \mathtt{5}\underline{\mathtt{c}}\mathtt{ebed} \quad \mathtt{5}\underline{\mathtt{e}}\mathtt{ebed}\ \right\} \\
\stackrel{\alpha=\mathtt{b}}{\longleftarrow} \left\{\ \mathtt{3}\underline{\mathtt{e}}\mathtt{ebed} \quad \mathtt{4}\underline{\mathtt{b}}\mathtt{bced}\ \right\} \\
\stackrel{\alpha=\mathtt{c}}{\longleftarrow} \left\{\ \mathtt{2}\underline{\mathtt{d}}\mathtt{dbed} \quad \mathtt{2}\underline{\mathtt{d}}\mathtt{ebed} \quad \mathtt{4}\underline{\mathtt{c}}\mathtt{ebed} \quad \mathtt{3}\underline{\mathtt{a}}\mathtt{bced}\ \right\} \\
\stackrel{\alpha=\mathtt{d}}{\longleftarrow} \left\{\ \mathtt{2}\underline{\mathtt{c}}\mathtt{dbed} \quad \mathtt{2}\underline{\mathtt{c}}\mathtt{ebed}\ \right\}
\end{cases}
\tag{93}
$$

$$
\mathtt{4}\alpha\underline{\mathtt{e}}\mathtt{cec}
\begin{cases}
\stackrel{\alpha=\mathtt{b}}{\longleftarrow} \mathtt{4}\underline{\mathtt{b}}\mathtt{ecec} \\
\stackrel{\alpha=\mathtt{c}}{\longleftarrow} \mathtt{3}\underline{\mathtt{a}}\mathtt{ecec} \\
\mathtt{3}\alpha\mathtt{eba}\underline{\mathtt{d}} \\
\mathtt{4}\alpha\mathtt{eba}\underline{\mathtt{d}} \\
\mathtt{3}\alpha\mathtt{ebb}\underline{\mathtt{d}} \\
\mathtt{4}\alpha\mathtt{ebb}\underline{\mathtt{d}}
\end{cases}
\qquad * \tag{94}
$$

$$\mathtt{4}\alpha\underline{\mathtt{e}}\zeta\mathtt{ce}\begin{cases}\overset{\alpha=\mathtt{b}}{\leftarrow}\mathtt{4}\underline{\mathtt{b}}\mathtt{e}\zeta\mathtt{ce}\\ \overset{\alpha=\mathtt{c}}{\leftarrow}\mathtt{3}\underline{\mathtt{a}}\mathtt{e}\zeta\mathtt{ce} \quad *\\ \overset{\zeta=\mathtt{c}}{\leftarrow}\mathtt{5}\alpha\mathtt{ebe}\underline{\mathtt{a}}\end{cases} \tag{95}$$

$$\mathtt{4}\delta\mathtt{bc}\underline{\mathtt{a}}\alpha\begin{cases}\overset{\alpha=\mathtt{a}}{\leftarrow}\mathtt{5}\delta\mathtt{bca}\underline{\mathtt{d}}\\ \overset{\alpha=\mathtt{c}}{\leftarrow}\{\ \mathtt{2}\delta\mathtt{bca}\underline{\mathtt{b}}\quad \mathtt{3}\delta\mathtt{bca}\underline{\mathtt{b}}\ \}\\ \overset{\alpha=\mathtt{d}}{\leftarrow}\mathtt{1}\delta\mathtt{bca}\underline{\mathtt{b}}\\ \overset{\delta=\mathtt{b}}{\leftarrow}\mathtt{3}\underline{\mathtt{e}}\mathtt{eaa}\alpha\\ \overset{\delta=\mathtt{c}}{\leftarrow}\{\ \mathtt{4}\underline{\mathtt{c}}\mathtt{eaa}\alpha\quad \mathtt{2}\underline{\mathtt{a}}\mathtt{ddc}\alpha\quad \mathtt{2}\underline{\mathtt{a}}\mathtt{edc}\alpha\ \}\end{cases} \quad * \text{ for } \delta\in\{\mathtt{b},\mathtt{c},\mathtt{e}\} \tag{96}$$

$$\mathtt{4}\delta\mathtt{bb}\underline{\mathtt{d}}\alpha\begin{cases}\overset{\alpha=\mathtt{a}}{\leftarrow}\mathtt{5}\delta\mathtt{bbd}\underline{\mathtt{d}}\\ \overset{\alpha=\mathtt{c}}{\leftarrow}\{\ \mathtt{2}\delta\mathtt{bbd}\underline{\mathtt{b}}\quad \mathtt{3}\delta\mathtt{bbd}\underline{\mathtt{b}}\ \}\\ \overset{\alpha=\mathtt{d}}{\leftarrow}\mathtt{1}\delta\mathtt{bbd}\underline{\mathtt{b}}\\ \overset{\delta=\mathtt{b}}{\leftarrow}\mathtt{4}\underline{\mathtt{b}}\mathtt{bbd}\alpha\end{cases} \quad * \text{ for } \delta\in\{\mathtt{b},\mathtt{e}\} \tag{97}$$

$$\mathtt{4dcb}\underline{\mathtt{d}}\alpha\begin{cases}\overset{\alpha=\mathtt{a}}{\leftarrow}\mathtt{5dcbd}\underline{\mathtt{d}}\\ \overset{\alpha=\mathtt{c}}{\leftarrow}\{\ \mathtt{2dcbd}\underline{\mathtt{b}}\quad \mathtt{3dcbd}\underline{\mathtt{b}}\ \}\\ \overset{\alpha=\mathtt{d}}{\leftarrow}\mathtt{1dcbd}\underline{\mathtt{b}}\end{cases} \tag{98}$$

$$\mathtt{5}\delta\mathtt{ba}\underline{\mathtt{d}}\alpha\leftarrow\begin{cases}\overset{\alpha=\mathtt{c}}{\leftarrow}\{\ \mathtt{3}\delta\mathtt{bad}\underline{\mathtt{d}}\quad \mathtt{4}\delta\mathtt{bad}\underline{\mathtt{d}}\ \}\\ \overset{\delta=\mathtt{b}}{\leftarrow}\mathtt{4}\underline{\mathtt{b}}\mathtt{bed}\alpha\\ \overset{\delta=\mathtt{c}}{\leftarrow}\mathtt{3}\underline{\mathtt{a}}\mathtt{bed}\alpha\end{cases} \quad * \text{ for } \delta\in\{\mathtt{b},\mathtt{c},\mathtt{e}\} \tag{99}$$

$$\mathtt{5}\gamma\mathtt{ca}\underline{\mathtt{d}}\alpha\begin{cases}\overset{\alpha=\mathtt{c}}{\leftarrow}\begin{cases}\mathtt{3}\gamma\mathtt{cad}\underline{\mathtt{d}}\\ \mathtt{4}\gamma\mathtt{cad}\underline{\mathtt{d}}\end{cases}\\ \overset{\gamma=\mathtt{a}}{\leftarrow}\{\ \mathtt{5}\underline{\mathtt{c}}\mathtt{aed}\alpha\quad \mathtt{5}\underline{\mathtt{e}}\mathtt{aed}\alpha\ \} \quad *\\ \overset{\gamma=\mathtt{b}}{\leftarrow}\{\ \mathtt{3}\underline{\mathtt{e}}\mathtt{aed}\alpha\quad \mathtt{4}\underline{\mathtt{b}}\mathtt{ebd}\alpha\ \}\\ \overset{\gamma=\mathtt{c}}{\leftarrow}\{\ \mathtt{4}\underline{\mathtt{c}}\mathtt{aed}\alpha\quad \mathtt{3}\underline{\mathtt{a}}\mathtt{ebd}\alpha\ \}\end{cases} \tag{100}$$

$$\mathtt{5da}\underline{\mathtt{d}}\alpha\leftarrow\{\ \mathtt{3cdad}\underline{\mathtt{d}}\quad \mathtt{4dad}\underline{\mathtt{d}}\ \} \tag{101}$$

AIRR of length 6

$$\mathtt{4}\alpha\underline{\mathtt{e}}\mathtt{cc}\epsilon\delta\begin{cases}\overset{\epsilon=\mathtt{a}}{\leftarrow}\begin{cases}\overset{\delta=\mathtt{a}}{\leftarrow}\mathtt{1}\alpha\mathtt{ebec}\underline{\mathtt{c}}\\ \overset{\delta=\mathtt{b}}{\leftarrow}\mathtt{4}\alpha\mathtt{ebec}\underline{\mathtt{a}}\\ \overset{\delta=\mathtt{c}}{\leftarrow}\mathtt{2}\alpha\mathtt{ebec}\underline{\mathtt{e}}\\ \overset{\delta=\mathtt{e}}{\leftarrow}\mathtt{5}\alpha\mathtt{ebec}\underline{\mathtt{b}}\end{cases}\\ \overset{\epsilon=\mathtt{e},\delta=\mathtt{c}}{\leftarrow}\{\ \mathtt{3}\alpha\mathtt{ebea}\underline{\mathtt{d}}\quad \mathtt{4}\alpha\mathtt{ebea}\underline{\mathtt{d}}\ \}\\ \overset{\alpha=\mathtt{b}}{\leftarrow}\mathtt{4}\underline{\mathtt{b}}\mathtt{ecc}\epsilon\delta\\ \overset{\alpha=\mathtt{c}}{\leftarrow}\mathtt{3}\underline{\mathtt{a}}\mathtt{ecc}\epsilon\delta\end{cases} \tag{102}$$

The following are probably not needed

Equations (103)-(108) can be obtained directly or from (34) by splitting up this by α and ζ and ϵ and adding the extra symbol δ and continuing the backward search algorithm.

$$2\alpha\underline{\mathtt{a}}\gamma\mathtt{c}\delta\begin{cases}\overset{\alpha=\mathtt{b}}{\longleftarrow}\left\{\ \mathtt{1\underline{d}a}\gamma\mathtt{c}\delta\quad\mathtt{1\underline{e}a}\gamma\mathtt{c}\delta\ \right\}\\ \overset{\gamma=\mathtt{d},\alpha=\mathtt{b}}{\longleftarrow}\left\{\ \mathtt{1\underline{d}dac}\delta\quad\mathtt{1\underline{e}dac}\delta\ \right\}\\ \overset{\gamma=\mathtt{e}}{\longleftarrow}\begin{cases}\overset{\alpha=\mathtt{b}}{\longleftarrow}\left\{\ \mathtt{4\underline{b}ecd}\delta\quad\mathtt{4\underline{b}eed}\delta\quad\mathtt{4\underline{b}eac}\delta\ \right\}\\ \overset{\alpha=\mathtt{c}}{\longleftarrow}\left\{\ \mathtt{3\underline{a}ecd}\delta\quad\mathtt{3\underline{a}eed}\delta\quad\mathtt{3\underline{a}eac}\delta\ \right\}\\ \overset{\delta=\mathtt{b}}{\longleftarrow}\left\{\ \mathtt{4}\alpha\mathtt{aad\underline{a}}\quad\mathtt{4}\alpha\mathtt{edd\underline{a}}\ \right\}\\ \overset{\delta=\mathtt{c}}{\longleftarrow}\left\{\begin{array}{l}\mathtt{2}\alpha\mathtt{aad\underline{e}}\quad\mathtt{2}\alpha\mathtt{aad\underline{b}}\quad\mathtt{3}\alpha\mathtt{aad\underline{b}}\quad\mathtt{2}\alpha\mathtt{edd\underline{e}}\\ \mathtt{2}\alpha\mathtt{edd\underline{b}}\quad\mathtt{3}\alpha\mathtt{edd\underline{b}}\end{array}\right\}\end{cases}\end{cases}\quad * \qquad (103)$$

$$2\alpha\underline{\mathtt{c}}\gamma\mathtt{ac}\begin{cases}\overset{\alpha=\mathtt{b}}{\longleftarrow}\left\{\ \mathtt{1\underline{d}c}\gamma\mathtt{ac}\quad\mathtt{1\underline{e}c}\gamma\mathtt{ac}\ \right\}\\ \overset{\gamma=\mathtt{d}}{\longleftarrow}\begin{cases}\overset{\alpha=\mathtt{a}}{\longleftarrow}\left\{\ \mathtt{4\underline{e}ccac}\quad\mathtt{5\underline{c}adac}\quad\mathtt{5\underline{e}adac}\quad\mathtt{4\underline{e}ecac}\ \right\}\\ \overset{\alpha=\mathtt{b}}{\longleftarrow}\left\{\begin{array}{l}\mathtt{1\underline{d}aaac}\quad\mathtt{1\underline{e}aaac}\quad\mathtt{4\underline{b}cdac}\quad\mathtt{1\underline{d}abac}\\ \mathtt{1\underline{e}abac}\quad\mathtt{3\underline{e}adac}\end{array}\right\}\\ \overset{\alpha=\mathtt{c}}{\longleftarrow}\left\{\ \mathtt{3\underline{a}cdac}\quad\mathtt{4\underline{c}adac}\ \right\}\\ \longleftarrow\left\{\ \mathtt{3}\alpha\mathtt{cdd\underline{d}}\quad\mathtt{4}\alpha\mathtt{cdd\underline{d}}\quad\mathtt{3}\alpha\mathtt{edd\underline{d}}\quad\mathtt{4}\alpha\mathtt{edd\underline{d}}\ \right\}\end{cases}\end{cases}\quad * \qquad (104)$$

$$2\alpha\underline{\mathtt{c}}\gamma\mathtt{c}\delta\begin{cases}\overset{\alpha=\mathtt{b}}{\longleftarrow}\left\{\ \mathtt{1\underline{d}c}\gamma\mathtt{c}\delta\quad\mathtt{1\underline{e}c}\gamma\mathtt{c}\delta\ \right\}\\ \overset{\gamma=\mathtt{d}}{\longleftarrow}\begin{cases}\overset{\alpha=\mathtt{a}}{\longleftarrow}\left\{\ \mathtt{4\underline{e}ccc}\delta\quad\mathtt{5\underline{c}cdc}\delta\quad\mathtt{4\underline{e}ecc}\delta\ \right\}\\ \overset{\alpha=\mathtt{b}}{\longleftarrow}\left\{\begin{array}{l}\mathtt{1\underline{d}aac}\delta\quad\mathtt{1\underline{e}aac}\delta\quad\mathtt{1\underline{d}abc}\delta\quad\mathtt{1\underline{e}abc}\delta\\ \mathtt{3\underline{e}cdc}\delta\end{array}\right\}\\ \overset{\alpha=\mathtt{c}}{\longleftarrow}\mathtt{4\underline{c}cdc}\delta\\ \overset{\delta=\mathtt{c}}{\longleftarrow}\left\{\ \mathtt{2}\alpha\mathtt{cdb\underline{e}}\quad\mathtt{2}\alpha\mathtt{edb\underline{e}}\ \right\}\end{cases}\\ \overset{\delta=\mathtt{b}}{\longleftarrow}\left\{\ \mathtt{4}\alpha\mathtt{cdb\underline{a}}\quad\mathtt{4}\alpha\mathtt{edb\underline{a}}\ \right\}\\ \overset{\gamma=\mathtt{e}}{\longleftarrow}\begin{cases}\overset{\alpha=\mathtt{c}}{\longleftarrow}\left\{\ \mathtt{2}\alpha\mathtt{cad\underline{e}}\quad\mathtt{2}\alpha\mathtt{cad\underline{b}}\quad\mathtt{3}\alpha\mathtt{cad\underline{b}}\ \right\}\\ \overset{\delta=\mathtt{b}}{\longleftarrow}\mathtt{4}\alpha\mathtt{cad\underline{a}}\end{cases}\end{cases}\quad * \qquad (105)$$

$$2\alpha\underline{\mathtt{c}}\gamma\mathtt{d}\delta\begin{cases}\overset{\alpha=\mathtt{b}}{\longleftarrow}\left\{\ \mathtt{1\underline{d}c}\gamma\mathtt{d}\delta\quad\mathtt{1\underline{e}c}\gamma\mathtt{d}\delta\ \right\}\\ \overset{\gamma=\mathtt{d}}{\longleftarrow}\begin{cases}\overset{\alpha=\mathtt{a}}{\longleftarrow}\left\{\ \mathtt{4\underline{e}ccd}\delta\quad\mathtt{4\underline{e}ecd}\delta\ \right\}\\ \overset{\alpha=\mathtt{b}}{\longleftarrow}\left\{\begin{array}{l}\mathtt{1\underline{d}aad}\delta\quad\mathtt{1\underline{e}aad}\delta\quad\mathtt{1\underline{d}abd}\delta\quad\mathtt{1\underline{e}abd}\delta\\ \mathtt{4\underline{b}cdd}\delta\end{array}\right\}\\ \overset{\alpha=\mathtt{c}}{\longleftarrow}\mathtt{3\underline{a}cdd}\delta\end{cases}\end{cases}\qquad (106)$$

$$\mathtt{2}\alpha\underline{\mathtt{d}}\gamma\mathtt{ac} \begin{cases} \overset{\alpha=\mathtt{b}}{\leftarrow} \left\{ \mathtt{1}\underline{\mathtt{d}}\mathtt{d}\gamma\mathtt{ac} \quad \mathtt{1}\underline{\mathtt{e}}\mathtt{d}\gamma\mathtt{ac} \right\} \\ \overset{\gamma=\mathtt{d}}{\leftarrow} \begin{cases} \mathtt{2}\alpha\mathtt{cbc}\underline{\mathtt{e}} \\ \overset{\alpha=\mathtt{b}}{\leftarrow} \left\{ \begin{matrix} \mathtt{1}\underline{\mathtt{d}}\mathtt{caac} \quad \mathtt{1}\underline{\mathtt{e}}\mathtt{caac} \quad \mathtt{4}\underline{\mathtt{b}}\mathtt{ccac} \quad \mathtt{4}\underline{\mathtt{b}}\mathtt{cbac} \\ \mathtt{4}\underline{\mathtt{b}}\mathtt{cecc} \end{matrix} \right\} \\ \overset{\alpha=\mathtt{c}}{\leftarrow} \left\{ \mathtt{3}\underline{\mathtt{a}}\mathtt{ccac} \quad \mathtt{3}\underline{\mathtt{a}}\mathtt{cbac} \quad \mathtt{3}\underline{\mathtt{a}}\mathtt{cecc} \right\} \end{cases} \end{cases} \quad * \tag{107}$$

$$\mathtt{2}\alpha\underline{\mathtt{d}}\gamma\mathtt{d} \begin{cases} \overset{\alpha=\mathtt{b}}{\leftarrow} \left\{ \mathtt{1}\underline{\mathtt{d}}\mathtt{d}\gamma\mathtt{d} \quad \mathtt{1}\underline{\mathtt{e}}\mathtt{d}\gamma\mathtt{d} \right\} \\ \overset{\gamma=\mathtt{d}}{\leftarrow} \begin{cases} \overset{\alpha=\mathtt{b}}{\leftarrow} \left\{ \mathtt{1}\underline{\mathtt{d}}\mathtt{cad} \quad \mathtt{4}\underline{\mathtt{b}}\mathtt{ccd} \quad \mathtt{1}\underline{\mathtt{e}}\mathtt{cad} \quad \mathtt{4}\underline{\mathtt{b}}\mathtt{cbd} \right\} \\ \overset{\alpha=\mathtt{c}}{\leftarrow} \left\{ \mathtt{3}\underline{\mathtt{a}}\mathtt{ccd} \quad \mathtt{3}\underline{\mathtt{a}}\mathtt{cbd} \right\} \end{cases} \end{cases} \tag{108}$$

$$\mathtt{2}\alpha\underline{\mathtt{a}}\mathtt{ae} \begin{cases} \overset{\alpha=\mathtt{a}}{\leftarrow} \left\{ \mathtt{4}\underline{\mathtt{e}}\mathtt{cce} \quad \mathtt{4}\underline{\mathtt{e}}\mathtt{ece} \quad \mathtt{5}\underline{\mathtt{c}}\mathtt{ade} \quad \mathtt{5}\underline{\mathtt{e}}\mathtt{ade} \right. \\ \overset{\alpha=\mathtt{b}}{\leftarrow} \left\{ \mathtt{1}\underline{\mathtt{d}}\mathtt{abe} \quad \mathtt{1}\underline{\mathtt{e}}\mathtt{abe} \quad \mathtt{3}\underline{\mathtt{e}}\mathtt{ade} \quad \mathtt{4}\underline{\mathtt{b}}\mathtt{cde} \right. \\ \overset{\alpha=\mathtt{c}}{\leftarrow} \left\{ \mathtt{3}\underline{\mathtt{a}}\mathtt{cde} \quad \mathtt{4}\underline{\mathtt{c}}\mathtt{ade} \right. \end{cases} \tag{109}$$

$$\mathtt{2}\alpha\underline{\mathtt{a}}\mathtt{de} \left\{ \overset{\alpha=\mathtt{b}}{\leftarrow} \left\{ \mathtt{1}\underline{\mathtt{d}}\mathtt{dae} \quad \mathtt{1}\underline{\mathtt{e}}\mathtt{dae} \right. \right. \tag{110}$$

$$\mathtt{2}\alpha\underline{\mathtt{a}}\mathtt{ee} \begin{cases} \overset{\alpha=\mathtt{b}}{\leftarrow} \mathtt{4}\underline{\mathtt{b}}\mathtt{eae} \\ \overset{\alpha=\mathtt{c}}{\leftarrow} \mathtt{3}\underline{\mathtt{a}}\mathtt{eae} \end{cases} \tag{111}$$

$$\mathtt{2}\alpha\underline{\mathtt{b}}\mathtt{ae} \begin{cases} \overset{\alpha=\mathtt{a}}{\leftarrow} \left\{ \mathtt{5}\underline{\mathtt{c}}\mathtt{ece} \quad \mathtt{5}\underline{\mathtt{e}}\mathtt{ece} \right. \\ \overset{\alpha=\mathtt{b}}{\leftarrow} \left\{ \mathtt{3}\underline{\mathtt{e}}\mathtt{ece} \right. \\ \overset{\alpha=\mathtt{c}}{\leftarrow} \left\{ \mathtt{4}\underline{\mathtt{c}}\mathtt{ece} \right. \end{cases} \tag{112}$$

$$\mathtt{2}\alpha\underline{\mathtt{c}}\mathtt{ae} \begin{cases} \overset{\alpha=\mathtt{b}}{\leftarrow} \left\{ \mathtt{4}\underline{\mathtt{b}}\mathtt{cbe} \quad \mathtt{4}\underline{\mathtt{b}}\mathtt{cce} \right. \\ \overset{\alpha=\mathtt{c}}{\leftarrow} \left\{ \mathtt{3}\underline{\mathtt{a}}\mathtt{cbe} \quad \mathtt{3}\underline{\mathtt{a}}\mathtt{cce} \right. \end{cases} \tag{113}$$

$$\mathtt{2}\alpha\underline{\mathtt{c}}\mathtt{de} \begin{cases} \overset{\alpha=\mathtt{a}}{\leftarrow} \left\{ \mathtt{4}\underline{\mathtt{e}}\mathtt{cce} \quad \mathtt{4}\underline{\mathtt{e}}\mathtt{ece} \quad \mathtt{5}\underline{\mathtt{c}}\mathtt{ade} \quad \mathtt{5}\underline{\mathtt{e}}\mathtt{ade} \right. \\ \overset{\alpha=\mathtt{b}}{\leftarrow} \left\{ \mathtt{1}\underline{\mathtt{d}}\mathtt{aae} \quad \mathtt{1}\underline{\mathtt{d}}\mathtt{abe} \quad \mathtt{1}\underline{\mathtt{e}}\mathtt{aae} \quad \mathtt{1}\underline{\mathtt{e}}\mathtt{abe} \quad \mathtt{3}\underline{\mathtt{e}}\mathtt{ade} \quad \mathtt{4}\underline{\mathtt{b}}\mathtt{cde} \right. \\ \overset{\alpha=\mathtt{c}}{\leftarrow} \left\{ \mathtt{3}\underline{\mathtt{a}}\mathtt{cde} \quad \mathtt{4}\underline{\mathtt{c}}\mathtt{ade} \right. \end{cases} \tag{114}$$

$$\mathtt{2}\alpha\underline{\mathtt{d}}\mathtt{de} \begin{cases} \overset{\alpha=\mathtt{b}}{\leftarrow} \begin{cases} \mathtt{1}\underline{\mathtt{d}}\mathtt{cae} \\ \mathtt{1}\underline{\mathtt{e}}\mathtt{cae} \\ \mathtt{1}\underline{\mathtt{d}}\mathtt{dde} \\ \mathtt{1}\underline{\mathtt{e}}\mathtt{dde} \\ \mathtt{4}\underline{\mathtt{b}}\mathtt{cce} \\ \mathtt{4}\underline{\mathtt{b}}\mathtt{cbe} \end{cases} \\ \overset{\alpha=\mathtt{c}}{\leftarrow} \left\{ \mathtt{3}\underline{\mathtt{a}}\mathtt{cce} \quad \mathtt{3}\underline{\mathtt{a}}\mathtt{cbe} \right\} \\ \leftarrow \left\{ \mathtt{5}\alpha\mathtt{cc}\underline{\mathtt{b}} \quad \mathtt{5}\alpha\mathtt{cb}\underline{\mathtt{a}} \quad \mathtt{5}\alpha\mathtt{cb}\underline{\mathtt{b}} \right\} \end{cases} \quad * \tag{115}$$

$$2\alpha\underline{\mathtt{d}}\mathtt{ee} \overset{\alpha=\mathtt{b}}{\longleftarrow} \left\{ \mathtt{1\underline{d}dee} \quad \mathtt{1\underline{e}dee} \right\} \tag{116}$$

$$2\alpha\underline{\epsilon}\gamma\mathtt{a} \overset{\alpha=\mathtt{b}}{\longleftarrow} \left\{ \begin{array}{l} \mathtt{1\underline{d}}\epsilon\gamma\mathtt{e} \\ \mathtt{1\underline{e}}\epsilon\gamma\mathtt{e} \end{array} \right. \quad \text{for } \gamma \in \{\mathtt{d},\mathtt{e}\} \text{ and } \epsilon \in \{\mathtt{a},\mathtt{c},\mathtt{d}\} \tag{117}$$

****

Summary of the strategy to characterise all the IRR.

Every IRR(n) with $\mathtt{n} > \mathtt{3}$ can be obtained from IRR(3) by adding a symbol α at the pointer in the RHS and on the LHS and origin and finding all α for which reachability holds and continuing the computation on the right. Do one cycle to increase $\mathtt{n}$ by 1. Then only one cycle (119)-(121) is needed, and in(123) the $\ldots$ can go and the $-$ branch.

The IRR are (generally) an infinite set of relations $\mathtt{A} \to \mathtt{B} \to \mathtt{C}$ where $\mathtt{A}$, $\mathtt{B}$ and $\mathtt{C}$ are CS's satisfying... The important idea being followed up here is that they can be grouped together by truncation from either end and manipulated to some extent as groups. This results in an infinite set of IRR being representable in terms of a possibly finite number of patterns. Whether this number is always finite is an interesting question. The truncation of any IRR is done from one end of the string only, and naturally it is the same end for the 3 strings of the 3 CS's. This is shown as $\ldots$ indicating arbitrary removed symbols. In the following schematic representation, to give the relevant details only, I just give the following information: the position of the pointer in the string, with a _ indicating that it has been removed by truncation, and the $\ldots$ indicating which side has been truncated. The original length of the strings is always at least one longer that the truncated length but is not specified. At the end I put the length in parentheses i.e. the number of symbols remaining in the strings and the type of the IRR (RR,RL,LR or LL) introduced earlier. Here note that the truncation is done after the computation so that the pointer can reach either end. Therefore a pointer position that has been removed by truncation is always on the same end of the string as the $\ldots$. The results are rather cryptic but give a concise description of the general process. (Warning to the reader: this is almost certainly incomprehensible without study of some examples in the paper)

Starting with the IRR(3) we have the forms

$$\mathtt{1} \to \mathtt{3} \to \left\{ \begin{array}{l} \mathtt{0}\ \mathrm{RL} \\ \mathtt{4}\ \mathrm{RR} \end{array} \right. \ (\mathtt{3}) \tag{118}$$

Those of type RL are truncated on the right, removing the rightmost symbols in the origin and the LHS (β) and the symbol in the RHS(γ) giving

$$\mathtt{1}\ldots \to \_\ldots \to \mathtt{0}\ldots\ (\mathtt{2}). \tag{119}$$

Note here the pointer at position 3 is now truncated so it appears as an underscore. Now an arbitrary symbol is added on the left (called α in the examples), and the backward searching algorithm is applied to find new origins. This will depend on α and generally restricts its values. Also, it allows the computation on the right to proceed to either the end of the string to generate members of IRR(4). The schematic representation of this is to add 1 to the pointer positions in the triplet of CS's, and introduce new origin and final CS representations thus:

$$\left.\begin{array}{ll} + & \texttt{1}\ldots \\ - & \texttt{3}\ldots \end{array}\right\} \rightarrow \texttt{2}\ldots \rightarrow \texttt{\_}\ldots \rightarrow \texttt{1}\ldots \rightarrow \left\{\begin{array}{l} \texttt{0}\ldots\text{RL} \\ \texttt{4}\ldots\text{RR} \end{array}\right. \quad (\texttt{3}) \tag{120}$$

Next, the rules of type RR can be ignored (i.e. if the pointer reaches position 3 with the next move to the right), and the origins of type + i.e. with the search for new origins go in the expected direction (not changing ends) will be separated out for consideration and the others of type − will be ignored for now. Also the intermediate forms will be omitted giving

$$\texttt{1}\ldots \rightarrow \texttt{\_}\ldots \rightarrow \texttt{0}\ldots \;\; (\texttt{3}). \tag{121}$$

These rules can now be truncated on the right by one symbol to generate rules of type (119). The reasoning here was found in an attempt to establish the set of results in (119) by induction on the length of the rules. The problem with it was the above omission of rules obtained by backward searching leading to origins with the pointer at the opposite end of the string. Normally, such rules could be simply discarded because it implies that the LHS (the middle CS of the triplets) does not have a demonstration that it is reachable. However here because truncation has been done from the right, it is possible that if the next symbol that was removed is now replaced that an origin with the pointer in the expected position (1) can now be obtained.

Thus returning to the previously omitted results, omitting intermediate steps gives

$$\texttt{3}\ldots \rightarrow \texttt{\_}\ldots \rightarrow \texttt{0}\ldots \;\; (\texttt{3}) \tag{122}$$

Adding back on the right the symbols β to the origin and LHS, and γ to the RHS and tracing back the origins again and doing forward computations gives results of the form

$$\left.\begin{array}{ll} + & \texttt{1}\ldots \\ - & \texttt{4}\ldots \end{array}\right\} \rightarrow \texttt{3}\ldots \rightarrow \texttt{\_}\ldots \rightarrow \left\{\begin{array}{l} \texttt{0}\ldots\ \text{RL} \\ \texttt{5}\ldots\ \text{RR} \end{array}\right. \quad (\texttt{4}) \tag{123}$$

Because this resulted from a rule in (118), there will be no more arbitrary symbols to be added, so the ... go and the postion of the pointer at the LHS _ becomes 4, and searching for new origins now gives only results from the +

branch because the − branch gives no proof of the reachability of the LHS.

$$\mathtt{1} \to \mathtt{4} \to \begin{cases} \mathtt{0}\ \text{RL} \\ \mathtt{5}\ \text{RR} \end{cases} .(4) \tag{124}$$

****

Now consider trying starting to obtain the IRR(4) in a similar way. Consider the IRR (**??**) of type RL, applying the same argument after adding a new arbitrary symbol α_4 on the left and searching for new origins gives

$$\mathtt{1}\alpha_4\underline{\alpha_3}\mathtt{ac} \begin{cases} \overset{\alpha_4=\mathtt{a}}{\longleftarrow} \mathtt{2}\underline{\mathtt{d}}\alpha_3\mathtt{ac} \\ \overset{\alpha_4=\mathtt{c}}{\longleftarrow} \mathtt{2}\underline{\mathtt{a}}\alpha_3\mathtt{ac} \quad \text{for } \alpha_3 \in \{\mathtt{d},\mathtt{e}\} \\ \overset{\alpha_4=\mathtt{d}}{\longleftarrow} \mathtt{2}\underline{\mathtt{c}}\alpha_3\mathtt{ac} \end{cases} \tag{125}$$

and for the RHS's we get

$$\mathtt{4}\underline{\alpha_4}\mathtt{caa} \begin{cases} \overset{\alpha_4=\mathtt{a}}{\longrightarrow} \mathtt{3\_bcaa} \\ \overset{\alpha_4=\mathtt{c}}{\longrightarrow} \mathtt{2abdb\_} \\ \overset{\alpha_4=\mathtt{d}}{\longrightarrow} \mathtt{5\_ccaa} \end{cases} \tag{126}$$

so the IRR(4) generated are

$$\left.\begin{array}{l} \mathtt{2}\underline{\mathtt{d}}\alpha_3\mathtt{ac} \to \mathtt{1abc}\underline{\mathtt{c}} \to \mathtt{3\_bcaa} \\ \mathtt{2}\underline{\mathtt{a}}\alpha_3\mathtt{ac} \to \mathtt{1cbc}\underline{\mathtt{c}} \to \mathtt{2abdb\_} \\ \mathtt{2}\underline{\mathtt{c}}\alpha_3\mathtt{ac} \to \mathtt{1dbc}\underline{\mathtt{c}} \to \mathtt{5\_ccaa} \end{array}\right\} \text{for } \alpha_3 \in \{\mathtt{d},\mathtt{e}\} \tag{127}$$

Now we see that the reverse rule used in the derivation of (**??**) namely (**??**) is followed by the reverse rule (125) used in the derivation of the extension of (**??**) by one symbol i.e. (127). Furthermore (125) can be expressed unambiguously as "(**??**).2 with $\alpha_3\mathtt{ac}$" where $\alpha_3 \in \{\mathtt{d},\mathtt{e}\}$. Here of course the string it is appended with is where the pointer is in the LHS of (**??**).2.

We can describe this situation by saying (**??**) must be followed by (**??**).2 with $\alpha_3\mathtt{ac}$ where $\alpha_3 \in \{\mathtt{d},\mathtt{e}\}$. This statement can be made without reference to (**??**) the original rule to be extended, however the set of symbols α_4 giving an IRR of type LR or RL allowing extension could depend on which branch of (**??**) is involved. In this case the derivation of (126) is by $\mathtt{4}\underline{\mathtt{a}} \to \mathtt{3\_b}$ for $\alpha_4 = \mathtt{a}$ and by $\mathtt{4}\underline{\mathtt{d}} \to \mathtt{5\_c}$ for $\alpha_4 = \mathtt{d}$ and for $\alpha_4 = \mathtt{c}$ the IRR is unextendable being of type RR. I will now try to continue the analysis by bringing out as many of these relationships as possible without writing down the full IRR generated at each stage.

The results of this were rather disappointing, mostly they seemed to be trivially obtained directly from (**??**) or combinations of two of them "side by side". A better way of proceeding might be using forward auxiliary rules i.e.

single branches of ARR written in reverse. Likewise starting from the first results in (43) and (44), adding α on the left and searching for origins gives

$$2\alpha_4\underline{\mathtt{d}}\alpha_1\mathtt{e}\left\{\begin{array}{l} \overset{\alpha_4=\mathtt{b}}{\longleftarrow}\left\{\begin{array}{l}1\underline{\mathtt{d}}\mathtt{d}\alpha_1\mathtt{e}\\ 1\underline{\mathtt{e}}\mathtt{d}\alpha_1\mathtt{e}\end{array}\right.\\ \overset{\alpha_1=\mathtt{a}}{\longleftarrow} 5\alpha_4\mathtt{dc}\underline{\mathtt{b}} \\ \overset{\alpha_1=\mathtt{d}}{\longleftarrow}\left\{\begin{array}{l} 5\alpha_4\mathtt{cb}\underline{\mathtt{a}}\\ 5\alpha_4\mathtt{cb}\underline{\mathtt{b}}\\ 5\alpha_4\mathtt{cc}\underline{\mathtt{b}}\\ \overset{\alpha_4=\mathtt{b}}{\longleftarrow}\left\{\begin{array}{l}1\underline{\mathtt{d}}\mathtt{cae}\\1\underline{\mathtt{e}}\mathtt{cae}\\4\underline{\mathtt{b}}\mathtt{cce}\\4\underline{\mathtt{b}}\mathtt{cbe}\end{array}\right.\\ \overset{\alpha_4=\mathtt{c}}{\longleftarrow}\left\{\begin{array}{l}3\underline{\mathtt{a}}\mathtt{cce}\\3\underline{\mathtt{a}}\mathtt{cbe}\end{array}\right.\end{array}\right.\end{array}\right. \quad \text{for } \alpha_1\in\{\mathtt{d},\mathtt{e}\} \qquad (128)$$

(**??**) and (**??**) is followed by (**??**).2 with $\beta\mathtt{c}\alpha_2$ where $\alpha_2 \in \{\mathtt{a},\mathtt{b},\mathtt{c}\}$ and $\beta \in \{\mathtt{d},\mathtt{e}\}$. Equation (125) can be abbreviated to

$$1\alpha_4\underline{\alpha_3}\mathtt{a}\left\{\begin{array}{l} \overset{\alpha_4=\mathtt{a}}{\longleftarrow} 2\underline{\mathtt{d}}\alpha_3\mathtt{a}\\ \overset{\alpha_4=\mathtt{c}}{\longleftarrow} 2\underline{\mathtt{a}}\alpha_3\mathtt{a}\\ \overset{\alpha_4=\mathtt{d}}{\longleftarrow} 2\underline{\mathtt{c}}\alpha_3\mathtt{a}\end{array}\right. \quad \text{for } \alpha_3\in\{\mathtt{d},\mathtt{e}\}. \qquad (129)$$

This could be thought to be the shortest form of the rule on the basis that a single reverse step could potentially go in either direction depending on the symbols α_4 and $\mathtt{a}$ on either side of the pointer. But actually regardless of what symbol replaces the $\mathtt{a}$, a reverse step in that direction is not possible so the rule does not depend on the $\mathtt{a}$ on the right. Therefore the rule can be further simplified to

$$1\alpha_4\underline{\alpha_3}\left\{\begin{array}{l} \overset{\alpha_4=\mathtt{a}}{\longleftarrow} 2\underline{\mathtt{d}}\alpha_3\\ \overset{\alpha_4=\mathtt{c}}{\longleftarrow} 2\underline{\mathtt{a}}\alpha_3\\ \overset{\alpha_4=\mathtt{d}}{\longleftarrow} 2\underline{\mathtt{c}}\alpha_3\end{array}\right. \quad \text{for } \alpha_3\in\{\mathtt{d},\mathtt{e}\}. \qquad (130)$$

Now notice that (130) has exactly the same form as (**??**) with different names of the symbols, and they can be combined to

$$1\alpha_4\underline{\alpha_3}\left\{\begin{array}{l} \overset{\alpha_4=\mathtt{a}}{\longleftarrow} 2\underline{\mathtt{d}}\alpha_3\\ \overset{\alpha_4=\mathtt{c}}{\longleftarrow} 2\underline{\mathtt{a}}\alpha_3\\ \overset{\alpha_4=\mathtt{d}}{\longleftarrow} 2\underline{\mathtt{c}}\alpha_3\end{array}\right. \quad \text{for } \alpha_3\in\{\mathtt{a},\mathtt{b},\mathtt{c},\mathtt{d},\mathtt{e}\}. \qquad (131)$$

In this form the result is explicitly independent of α_3. Eliminating this gives just (**??**).2.

****

Now consider the first few steps of the IRR $\mathtt{x} = \mathtt{1} \to \mathtt{n} \to \mathtt{0}$ summarised as $\mathtt{1} \to \mathtt{2}$ such that after this point the pointer never again reaches 2 before it reaches the n. Then this forms the auxiliary rule (AR) (the reverse of one branch of the ARR) y associated with the derivation of $\mathtt{x} \in \mathrm{IRR}(\mathtt{n})$ because the remainder of the computation in x has a subset of CS's in the sequence $\mathtt{z} = \mathtt{2} \to \mathtt{n} \to \mathtt{1}$ where the CS 1 (first time the pointer gets to position 1) is the first CS 1 following the CS n and there is no other CS 2 or n between the CS 2 and the CS n (see(**??**)). z has a redundant symbol on the left applied to a rule of length $\mathtt{n}-\mathtt{1}$ of the form $\mathtt{1} \to \mathtt{n}-\mathtt{1} \to \mathtt{0}$ which is $\in \mathrm{IRR}(\mathtt{n}-\mathtt{1})$ by Lemma **??**. Any AR can be placed in this context.

Now add to y a single symbol β on the right giving another rule $\mathtt{y}\beta$ of the form $\mathtt{1} \to \mathtt{2}$ of length $\mathtt{n}+\mathtt{1}$. Can this be an AR for an IRR of type RL of length $\mathtt{n}+\mathtt{1}$? This would require the pointer to reach $\mathtt{n}+\mathtt{1}$ then 0 but actually because the computation follows the path $\mathtt{x}\beta$ that only gets to n before reaching 0 the answer is no. Likewise for the addition of any other string in place of β.

This suggests finding out whether any rule r of the form $\mathtt{1} \to \mathtt{2}$ of length l is an or can be made to be an AR for some IRR(n) of type RL for some $\mathtt{n} \geq \mathtt{l}$ by adding some string of redundant symbols on the right if necessary. If so, the added string must be unique. Continuing the computation from r the pointer will eventually reach 0 or $\mathtt{l}+\mathtt{1}$ unless a stationary cycle is reached. In the first case, if the pointer does not reach 1 again before it reaches its first maximum point k then r is an AR for a member of IRR(k). If it does reach 1 again before the first maximum point, r and any of its extensions $\mathtt{r}\beta$ for any string β is not an AR but the search can start again from the new 1 CS. In the second case by adding a symbol on the right, and continuing the computation to eventually reach 0 or $\mathtt{l}+\mathtt{2}$ unless a stationary cycle is reached and the whole argument can be repeated.

This argument could of course be expressed in its mirror image form. It may not be necessary to do this explicitly. The case $\mathtt{l} = \mathtt{2}$ probably needs special attention.

The AR's (an infinite set) for TM1 in [2] are generated by the results in Tables 4 and 5, which are in turn related to ARR's of length 2 or 3. It is probably useful to try to formulate hypotheses similar to these based on the ARR's in this example and modify them as needed by attempting a proof by induction on the length of the rules.

A truncated auxiliary reverse rule (TARR) is an auxiliary reverse rule (ARR) with branches that do not lead to a proof of reachability in the new IRR to be derived deleted. For this to make sense, the direction as indicated by the position of the symbol called α must be known. I will use $\alpha_{\mathtt{n}}$ to indicate this α as distinct from earlier ones where the subscript n is the length of the rule that is obtained by adding it. This implies that a TARR is an ARR such that all branches that do not lead to origins with the pointer not where the α

is have been deleted.

The TARR (??) is irreducible (TAIRR) in the sense that all the symbols are now involved. In the result (125) the c was clearly not involved because the pointer never reached within one symbol of it. For TARR, a symbol at the end of the string x is involved if the pointer reaches the adjacent symbol and the subsequent calculations on those branches lead to at least one new origin. Attempting to shorten the rule by one by deleting the x would result in termination of the backward search algorithm and those origins would not be obtained. If however the pointer reaches the adjacent symbol to x and none of the subsequent calculations result in new origins, then x can be deleted along with those branches leading to a TARR of length shorter by 1 that than the original one. This definition allows the definition of a minimal length TARR i.e. a TAIRR.

****

2 An interesting relationship between the sets of hypotheses

It is a striking fact that the LHS corresponding to any origin, apart from its state, is the same between each of (??)-(??) and (??). For example the origin $5\ldots a\underline{d}$ results in the LHS $3\ldots ba$ in (??) and in $2\ldots ba$, $4\ldots ba$, and $5\ldots ba$ in (??)-(??). This has a simple explanation, with each derivation of an IRR being completed by using Lemma ??. The result

$$5a\underline{d} \to 4\underline{a}a \to 3\_ba \tag{132}$$

explains the result in (??) after putting any symbol on the left because this computation eventually goes to the right (to get an irreducible rule), and putting c on the left gives

$$5ca\underline{d} \to 3\underline{c}ba \to 2\_aba. \tag{133}$$

Because $3\underline{c}$ goes left this gives in the corresponding result in (??) after again putting any symbol on the left because the the pointer ends up on the right. Likewise $3\underline{b}$ and $3\underline{d}$ go left so b and d can be put on the left of (132) giving the other two results above. In all these results because the symbols are added on the left, their summaries keeping the rightmost two symbols are the same apart from the machine states.

More generally one can argue as follows where for each IRR mentioned, the origin and LHS only are specified, and greek letters represent arbitrary symbols, and n_1,n_2 and n_3 represent machine states.

Lemma 2.1. *Suppose (1) for some positive integer* $\mathtt{m}$ *an abbreviation of a member of IRR(*$\mathtt{m}$*) in triplet form is*

$$\mathtt{n}_1\underline{\alpha_1}\ldots\alpha_{\mathtt{r}}\ldots\rightarrow\mathtt{n}_2\beta_1\ldots\beta_{\mathtt{r}}\ldots\rightarrow\ldots \tag{134}$$

where $\mathtt{r}<\mathtt{m}$ *is another positive integer. The origin and LHS of the rule are shown abbreviated by deleting symbols from the right hand end so that they have length* $\mathtt{r}$ *then* (134) *in full must have either of the types*

$$\mathtt{1}\rightarrow\mathtt{m}\rightarrow\left\{\begin{array}{l}\mathtt{0}\\ \mathtt{1}\rightarrow\mathtt{m}+\mathtt{1}\end{array}\right. . \tag{135}$$

Therefore the movement of the pointer after the middle member of (134) *(the* $\mathtt{m}$*) must be left.*

(2) there is a right-moving symbol with state $\mathtt{n}_2$ *i.e. there are symbols* γ^* *and* δ *such that*

$$\mathtt{n}_2\underline{\gamma^*}\rightarrow\mathtt{n}_3\delta_{-} \tag{136}$$

and (3) in the related computation in which the rightmost symbol is changed to γ^**, the pointer reaches position 1 after it moved to just right of the* δ *i.e. after* (136) *was applied.*

Then a member of IRR($\mathtt{m}+\mathtt{1}$*) can be likewise abbreviated as*

$$\mathtt{n}_1\underline{\alpha_1}\ldots\alpha_{\mathtt{r}}\ldots\rightarrow\mathtt{n}_3\beta_1\ldots\beta_{\mathtt{r}}\ldots\rightarrow\ldots. \tag{137}$$

Condition (3) is satisfied if the pointer eventually reaches position 0. In this case the member of IRR($\mathtt{m}+\mathtt{1}$*) has the type RL.*

Note that the only difference between (134) *and* (137) *is that the state* $\mathtt{n}_2$ *is replaced by* $\mathtt{n}_3$*.*

Proof. The member of IRR($\mathtt{m}$) referred to in (134) can be written as

$$\mathtt{n}_1\underline{\alpha_1}\ldots\alpha_{\mathtt{r}}\gamma_1\ldots\gamma_{\mathtt{m-r-1}}\gamma_0\rightarrow\mathtt{n}_2\beta_1\ldots\beta_{\mathtt{r}}\gamma'_1\ldots\gamma'_{\mathtt{m-r-1}}\underline{\gamma'_0}\rightarrow\ldots \tag{138}$$

where $\gamma_1,\ldots\gamma_{\mathtt{m-r-1}}\gamma_0$ and $\gamma'_1\ldots\gamma'_{\mathtt{m-r-1}}\gamma'_0$ are two strings of symbols of length $\mathtt{m}-\mathtt{r}$. By Lemma **??**, the pointer for the first time reaches the γ'_0 in (138). This shows that

$$\gamma'_0=\gamma_0 \tag{139}$$

and (138) imposes no condition on γ_0, so one can choose $\gamma_0=\gamma^*$ in a new computation. Combining(136), (138) and(139) shows that

$$\mathtt{n}_1\underline{\alpha_1}\ldots\alpha_{\mathtt{r}}\gamma_1\ldots\gamma_{\mathtt{m-r-1}}\gamma^*\rightarrow\mathtt{n}_3\beta_1\ldots\beta_{\mathtt{r}}\gamma'_1\ldots\gamma'_{\mathtt{m-r-1}}\delta_{-} \tag{140}$$

where for the first time the pointer reaches just to the right of the δ so adding another arbitrary symbol ϵ on the right and continuing using (3) shows that

$$\mathtt{n}_1\underline{\alpha_1}\ldots\alpha_{\mathtt{r}}\gamma_1\ldots\gamma_{\mathtt{m-r-1}}\gamma^*\epsilon\rightarrow\mathtt{n}_3\beta_1\ldots\beta_{\mathtt{r}}\gamma'_1\ldots\gamma'_{\mathtt{m-r-1}}\delta\underline{\epsilon}\rightarrow\mathtt{n}_4\underline{\beta_1}\ldots\rightarrow\ldots \tag{141}$$

and this is a member of IRR($\mathtt{m}+\mathtt{1}$) regardless of at which end the pointer finishes. If this is abbreviated by truncation it gives (137). □

Lemma 2.2. *Suppose (1) for some positive integer* $\mathtt{m}$ *an abbreviation of a member* $\mathtt{x}$ *of IRR(*$\mathtt{m}$*) in triplet form (with no information on the RHS) is*

$$\mathtt{n}_1\underline{\alpha_1}\ldots\alpha_{\mathtt{r}}\ldots\to\mathtt{n}_2\beta_1\ldots\beta_{\mathtt{r}}\ldots\to\ldots \tag{142}$$

where $\mathtt{r}<\mathtt{m}$ *is another positive integer. The origin and LHS of the rule are shown abbreviated by deleting symbols from the right hand end so that they have length* $\mathtt{r}$ *and*

(2) if the symbol at the right hand end of the middle member of (142) *(say* γ'_0*) is replaced by say* γ^* *(giving computation* $\mathtt{y}$*) so that this matches the LHS of the following irreducible rule* $\mathtt{z}$ *(that must be* $\in$ *IRR(*$\mathtt{s}$*) because its LHS is clearly reachable)*

$$\mathtt{n}_2\gamma'_{\mathtt{m}-\mathtt{r}+1-\mathtt{s}}\ldots\gamma'_{\mathtt{m}-\mathtt{r}-1}\underline{\gamma^*}\to\mathtt{n}_3\delta_1\ldots\delta_{\mathtt{s}-}. \tag{143}$$

where $\mathtt{s}\leq\mathtt{m}-\mathtt{r}$ *and*

(3) in computation $\mathtt{y}$ *the pointer reaches position 1 after it was just right of* $\delta_{\mathtt{s}}$ *(note this depends on the next symbol)*

then the new computation $\mathtt{y}$ *generates a member of IRR(*$\mathtt{m}+\mathtt{1}$*) which can can be likewise abbreviated as*

$$\mathtt{n}_1\underline{\alpha_1}\ldots\alpha_{\mathtt{r}}\ldots\to\mathtt{n}_3\beta_1\ldots\beta_{\mathtt{r}}\ldots\to\ldots. \tag{144}$$

Condition (3) is satisfied if in $\mathtt{y}$ *the pointer eventually reaches position 0. In this case the member of IRR(*$\mathtt{m}+\mathtt{1}$*) has the type RL. If condition (3) fails,* $\mathtt{y}$ *generates a reducible rule so does not generate a member of IRR(*$\mathtt{m}+\mathtt{1}$*).*

Note that the only difference between (142) *and* (144) *is that the state* $\mathtt{n}_2$ *is replaced by* $\mathtt{n}_3$*.*

Proof. The IRR $\mathtt{x}$ can be written as

$$\mathtt{n}_1\underline{\alpha_1}\ldots\alpha_{\mathtt{r}}\gamma_1\ldots\gamma_{\mathtt{m}-\mathtt{r}-1}\gamma_0\to\mathtt{n}_2\beta_1\ldots\beta_{\mathtt{r}}\gamma'_1\ldots\gamma'_{\mathtt{m}-\mathtt{r}-1}\underline{\gamma'_0}\to\ldots \tag{145}$$

where $\gamma_1,\ldots\gamma_{\mathtt{m}-\mathtt{r}-1}\gamma_0$ and $\gamma'_1\ldots\gamma'_{\mathtt{m}-\mathtt{r}-1}\gamma'_0$ are two strings of symbols of length $\mathtt{m}-\mathtt{r}$. Then $\mathtt{x}$ must have either of the types where only the extreme pointer positions are shown in each left to right or right to left movement:

$$\mathtt{1}\to\mathtt{m}\to\begin{cases}\mathtt{0}\\ \mathtt{1}\to\mathtt{m}+\mathtt{1}\end{cases}. \tag{146}$$

Therefore the movement of the pointer after the middle member of (142) (the $\mathtt{m}$) must be left. By Lemma **??**, the pointer for the first time reaches the γ'_0 in (145). This shows that

$$\gamma'_0=\gamma_0 \tag{147}$$

and (145) imposes no condition on γ_0. Therefore replacing γ_0 by γ^* and combining (143), (145) and(147) shows that in computation $\mathtt{y}$

$$\mathtt{n}_1\underline{\alpha_1}\ldots\alpha_\mathtt{r}\gamma_1\ldots\gamma_{\mathtt{m-r-1}}\gamma^* \to \mathtt{n}_3\beta_1\ldots\beta_\mathtt{r}\gamma'_1\ldots\gamma'_{\mathtt{m-r-s}}\delta_1\ldots\delta_{\mathtt{s}\underline{-}}. \tag{148}$$

Here for the first time the pointer reaches just to the right of the $\delta_\mathtt{s}$ so adding another arbitrary symbol ϵ on the right and continuing shows that in $\mathtt{y}$

$$\mathtt{n}_1\underline{\alpha_1}\ldots\alpha_\mathtt{r}\gamma_1\ldots\gamma_{\mathtt{m-r-1}}\gamma^*\epsilon \to \mathtt{n}_3\beta_1\ldots\beta_\mathtt{r}\gamma'_1\ldots\gamma'_{\mathtt{m-r-s}}\delta_1\ldots\delta_\mathtt{s}\underline{\epsilon} \to \ldots. \tag{149}$$

demonstrating the reachability of its middle member (the LHS). Condition (3) implies that the rule generating its RHS is irreducible, hence computation $\mathtt{y}$ generates an IRR of length $\mathtt{m}+1$. If this is abbreviated by truncation it gives (144). □

****

3 Going back to pick up from the more complex cases extending the set of inductive hypotheses for IRR($\mathtt{n}$)

By applying the argument to obtain (**??**) again (ignoring the − branches of the ARR) to the origins in (**??**), it is easy to show that the new origins are all in at least one of (**??**),(**??**),(**??**), and (**??**). Therefore this set $\mathtt{S}$ of 60 SCS's is closed under a repetition of the argument used to obtain (**??**) above (ignoring the − branches of the ARR), essentially based on (**??**). That is, for each SCS $\mathtt{X}$ in $\mathtt{S}$, and for each symbol α there is a set of members $\mathtt{Y}\in\mathtt{S}$ obtained by applying the backward search algorithm for a single step with the pointer moving to end i.e. to the left or right if $\mathtt{X}$ has the ... on the right or left respectively.

For each $\mathtt{X}\in\mathtt{S}$, using (**??**), $\mathtt{Y}$ has at most 2 elements and for each $\mathtt{X}\in\mathtt{S}$ the set $\{\alpha|\mathtt{Y}\neq\emptyset\}$ can be read off directly from (**??**) for example, if $\mathtt{X}=\mathtt{1}\underline{\mathtt{d}}\mathtt{a}\ldots$ then $\mathtt{1}\alpha\underline{\mathtt{d}}\mathtt{a}\ldots$ has origins in one step if and only if $\alpha\in\{\mathtt{a},\mathtt{c},\mathtt{d}\}$. The single step requirement is related to the way in which the subset of origins for the IRR(4) in (**??**),(**??**),(**??**), and (**??**) were obtained.

This leads to the question, can all the IRR for a TM be included in abbreviations like (142) and (144)? If so, what is the largest value of $\mathtt{r}$ needed? or could it be infinite? This must be to do with ensuring that the induction argument works. If the induction argument had worked i.e. if a set of results similar to (**??**)-(**??**) could be reproduced exactly (ignoring the RHS's for now) by applying arguments similar to (**??**) i.e. without generating any extra results, then (**??**)-(**??**) would be established by induction. What happens if this doesn't quite work?

Can every member of IRR($\mathtt{m}+\mathtt{1}$) can also be obtained by Lemma 2.2 from some member of IRR($\mathtt{m}$)? Not in general because the rule it derives from is not in general an IRR. If so this would explain the general form of the set of results produced by the induction argument.

The member of IRR($\mathtt{m}$) referred to in (134) can have the form

$$\mathtt{1} \to \mathtt{k} \to \mathtt{2} \to \mathtt{m} \to \mathtt{0} \tag{150}$$

for any $\mathtt{k} \leq \mathtt{m} - \mathtt{1}$ where $\mathtt{k}$ is the rightmost position of the pointer before it reaches $\mathtt{2}$. This form would arise if the IRR of length $\mathtt{m}$ had been obtained from one of length $\mathtt{m}-\mathtt{1}$ of the form $\mathtt{1} \to \mathtt{m}-\mathtt{1} \to \mathtt{0}$ by addition of a symbol to the left and searching for new origins. The simplest non-trivial case is when $\mathtt{k} = \mathtt{3}$ and this happens when in the search for origins, the pointer goes in the unexpected $-$ direction once.

****

$$\mathtt{4}\alpha\underline{\mathtt{b}}\mathtt{ca} \dashleftarrow \left\{ \begin{array}{l} \mathtt{3}\alpha\mathtt{bb}\underline{\mathtt{c}} \\ \mathtt{1}\alpha\mathtt{bb}\underline{\mathtt{c}} \\ \mathtt{5}\alpha\mathtt{ea}\underline{\mathtt{d}} \\ \overset{\alpha=\mathtt{a}}{\longleftarrow} \left\{ \begin{array}{l} \mathtt{2}\underline{\mathtt{d}}\mathtt{dba} \\ \mathtt{2}\underline{\mathtt{d}}\mathtt{eba} \\ \mathtt{5}\underline{\mathtt{c}}\mathtt{eba} \\ \mathtt{5}\underline{\mathtt{e}}\mathtt{eba} \end{array} \right. \\ \overset{\alpha=\mathtt{b}}{\longleftarrow} \mathtt{3}\underline{\mathtt{e}}\mathtt{eba} \\ \overset{\alpha=\mathtt{c}}{\longleftarrow} \left\{ \begin{array}{l} \mathtt{2}\underline{\mathtt{a}}\mathtt{dba} \\ \mathtt{2}\underline{\mathtt{a}}\mathtt{eba} \\ \mathtt{4}\underline{\mathtt{c}}\mathtt{eba} \end{array} \right. \\ \overset{\alpha=\mathtt{d}}{\longleftarrow} \left\{ \begin{array}{l} \mathtt{2}\underline{\mathtt{c}}\mathtt{dba} \\ \mathtt{2}\underline{\mathtt{c}}\mathtt{eba} \end{array} \right. \end{array} \right. \tag{151}$$

$$\mathtt{4}\alpha\underline{\mathtt{b}}\mathtt{ce} \dashleftarrow \left\{ \begin{array}{l} \mathtt{5}\alpha\mathtt{bb}\underline{\mathtt{a}} \\ \mathtt{5}\alpha\mathtt{bb}\underline{\mathtt{b}} \\ \overset{\alpha=\mathtt{a}}{\longleftarrow} \left\{ \begin{array}{l} \mathtt{2}\underline{\mathtt{d}}\mathtt{dbe} \\ \mathtt{2}\underline{\mathtt{d}}\mathtt{ebe} \\ \mathtt{5}\underline{\mathtt{c}}\mathtt{ebe} \\ \mathtt{5}\underline{\mathtt{e}}\mathtt{ebe} \end{array} \right. \\ \overset{\alpha=\mathtt{b}}{\longleftarrow} \mathtt{3}\underline{\mathtt{e}}\mathtt{ebe} \\ \overset{\alpha=\mathtt{c}}{\longleftarrow} \left\{ \begin{array}{l} \mathtt{2}\underline{\mathtt{a}}\mathtt{dbe} \\ \mathtt{2}\underline{\mathtt{a}}\mathtt{ebe} \\ \mathtt{4}\underline{\mathtt{c}}\mathtt{ebe} \end{array} \right. \\ \overset{\alpha=\mathtt{d}}{\longleftarrow} \left\{ \begin{array}{l} \mathtt{2}\underline{\mathtt{c}}\mathtt{dbe} \\ \mathtt{2}\underline{\mathtt{c}}\mathtt{ebe} \end{array} \right. \end{array} \right. \tag{152}$$

The results (151) and (152) were obtained in the same way from (**??**) and so are not associated with members of IRR(3).

****

4 Rearrangement of the calculations

The above is a brief description of how the IRR can all be obtained in a fairly efficient manner. However when doing this you will notice that the results can be grouped into sets that are similar, and that by making use of more than one α parameter (I will use α_1, α_2 etc.) the presentation and calculations can be made even more efficient because the parts of the calculations that only have have to be done once in each set are in fact done once only. Also only the IRR that can be extended (types LR and RL) will be explicitly mentioned (EIRR), the others can be easily obtained later if required. There is one further important point when organising these results: the central CS in the triplet of CS's representing an IRR together with proof of the reachability of its LHS is extended by one symbol at each elongation step from length 2 to length 3, then from length 3 to length 4 etc. and the left and right hand CS's vary in a much more complicated way between elongation steps and so are not useful for this purpose. Thus this CS (the B above) should be used to index all the sets of calculations i.e. all the results will be finally sorted by their B CS. For this reason I will put B into the subsection heading in which the corresponding triples $\mathtt{A} \to \mathtt{B} \to \mathtt{C}$ representing the IRR are derived. This also has the advantage that each IRR need only be represented by $\mathtt{C} \to \mathtt{A}$ because the B can be inserted by the reader.

4.1 $\mathtt{B} = \mathtt{2}\underline{\alpha_1}\mathtt{a}$

Starting from $\mathtt{3\underline{c}} \to \mathtt{2\_a}$ adding the α_1 gives

$$\mathtt{3}\alpha_1\mathtt{\underline{c}} \to \mathtt{2}\underline{\alpha_1}\mathtt{a} \left\{ \begin{array}{l} \stackrel{\alpha_1=\mathtt{a}}{\to} \mathtt{2db\_} \\ \stackrel{\alpha_1=\mathtt{c}}{\to} \mathtt{2ab\_} \\ \stackrel{\alpha_1=\mathtt{d}}{\to} \mathtt{2cb\_} \end{array} \right\} \tag{153}$$

as the representation of the extension of $\mathtt{3\underline{c}} \to \mathtt{2\_a}$ to IRR of length 2.

4.2 $\mathtt{B} = \mathtt{2}\underline{\alpha_1}\mathtt{a}\alpha_2$

Next adding α_2 on the right and searching for the new origins gives

$$\mathtt{3}\alpha_1\mathtt{\underline{c}}\alpha_2 \leftarrow \left\{ \begin{array}{l} \stackrel{\alpha_2=\mathtt{a}}{\leftarrow} \mathtt{1}\alpha_1\mathtt{c\underline{c}} \\ \stackrel{\alpha_2=\mathtt{b}}{\leftarrow} \mathtt{4}\alpha_1\mathtt{c\underline{a}} \\ \stackrel{\alpha_2=\mathtt{c}}{\leftarrow} \mathtt{2}\alpha_1\mathtt{c\underline{e}} \\ \stackrel{\alpha_2=\mathtt{e}}{\leftarrow} \mathtt{5}\alpha_1\mathtt{c\underline{b}} \\ \dots \end{array} \right. \tag{154}$$

Here I will use the convention that origins not leading to any proofs of reachability will be omitted and replaced by dots, i.e. the reverse rules are truncated so that their type is $\neg\mathtt{M}$. Involved in (153) is the function that maps α_1 to the first letter on its RHS's which I will denote by $\mathtt{f}_1$ thus

$$\begin{array}{c|c} \mathtt{x} & \mathtt{f}_1(\mathtt{x}) \\ \hline \mathtt{a} & \mathtt{d} \\ \mathtt{c} & \mathtt{a} \\ \mathtt{d} & \mathtt{c} \end{array} \tag{155}$$

This will make the presentation far more compact. Now the derivation of the IRR(3) of types LR and RL from $\mathtt{2\_a}$ can be completed as follows

$$\begin{array}{l} \mathtt{1}\alpha_1\mathtt{c}\underline{\mathtt{c}} \to \mathtt{3}\alpha_1\underline{\mathtt{c}}\mathtt{a} \to \mathtt{2}\underline{\alpha_1}\mathtt{aa} \to \mathtt{2f}_1(\alpha_1)\mathtt{b}\underline{\mathtt{a}} \to \mathtt{1f}_1(\alpha_1)\mathtt{bc\_} \\ \mathtt{4}\alpha_1\mathtt{c}\underline{\mathtt{a}} \to \mathtt{3}\alpha_1\underline{\mathtt{c}}\mathtt{b} \to \mathtt{2}\underline{\alpha_1}\mathtt{ab} \to \mathtt{2f}_1(\alpha_1)\mathtt{b}\underline{\mathtt{b}} \to \mathtt{3f}_1(\alpha_1)\mathtt{bc\_} \\ \mathtt{2}\alpha_1\mathtt{c}\underline{\mathtt{e}} \to \mathtt{3}\alpha_1\underline{\mathtt{c}}\mathtt{c} \to \mathtt{2}\alpha_1\mathtt{ac} \to \mathtt{2f}_1(\alpha_1)\mathtt{b}\underline{\mathtt{c}} \to \mathtt{1f}_1(\alpha_1)\mathtt{bd\_} \\ \mathtt{5}\alpha_1\mathtt{c}\underline{\mathtt{b}} \to \mathtt{3}\alpha_1\underline{\mathtt{c}}\mathtt{e} \to \mathtt{2}\underline{\alpha_1}\mathtt{ae} \to \mathtt{2f}_1(\alpha_1)\mathtt{b}\underline{\mathtt{e}} \to \mathtt{4}\underline{\mathtt{f}_1(\alpha_1)}\mathtt{cc} \end{array} \tag{156}$$

where the computation in (156).4 is not complete and leads to

$$\mathtt{4}\underline{\mathtt{f}_1(\alpha_1)}\mathtt{cc} \left\{ \begin{array}{l} \stackrel{\alpha_1=\mathtt{a}}{\to} \mathtt{5\_ccc} \\ \stackrel{\alpha_1=\mathtt{c}}{\to} \mathtt{3\_bcc} \\ \stackrel{\alpha_1=\mathtt{d}}{\to} \mathtt{1abd\_} \end{array} \right. \tag{157}$$

The result for the extendable IRR(3) can be summarised as

$$\begin{array}{ccc} \alpha_1 & \alpha_2 & \mathtt{C} \to \mathtt{B} \to \mathtt{A} \\ \hline \{\mathtt{a},\mathtt{c},\mathtt{d}\} & \mathtt{a} & \mathtt{1}\alpha_1\mathtt{c}\underline{\mathtt{c}} \to \mathtt{2}\underline{\alpha_1}\mathtt{a}\alpha_2 \to \mathtt{1f}_1(\alpha_1)\mathtt{bc\_} \\ \{\mathtt{a},\mathtt{c},\mathtt{d}\} & \mathtt{b} & \mathtt{4}\alpha_1\mathtt{c}\underline{\mathtt{a}} \to \mathtt{2}\underline{\alpha_1}\mathtt{a}\alpha_2 \to \mathtt{3f}_1(\alpha_1)\mathtt{bc\_} \\ \{\mathtt{a},\mathtt{c},\mathtt{d}\} & \mathtt{c} & \mathtt{2}\alpha_1\mathtt{c}\underline{\mathtt{e}} \to \mathtt{2}\underline{\alpha_1}\mathtt{a}\alpha_2 \to \mathtt{1f}_1(\alpha_1)\mathtt{bd\_} \\ \mathtt{d} & \mathtt{e} & \mathtt{5dc}\underline{\mathtt{b}} \to \mathtt{2}\underline{\mathtt{d}}\mathtt{ae} \to \mathtt{1abd\_} \end{array} \tag{158}$$

4.3 $\mathtt{B} = \mathtt{2}\underline{\alpha_1}\mathtt{a}\alpha_2\alpha_3$

Now use the method again to derive the IRR(4) that are extendable, from these extendable IRR(3). This will start with adding α_3 on the right in these 4 cases separately (because it seems that they cannot be easily combined). From (158).1

$$\mathtt{1}\alpha_1\mathtt{c}\underline{\mathtt{c}}\alpha_3 \leftarrow \mathtt{2}\alpha_1\underline{\mathtt{a}}\mathtt{c}\alpha_3 \stackrel{\alpha_1=\mathtt{b}}{\leftarrow} \left\{ \begin{array}{l} \mathtt{1}\underline{\mathtt{d}}\mathtt{ac}\alpha_3 \\ \mathtt{1}\underline{\mathtt{e}}\mathtt{ac}\alpha_3 \end{array} \right. \tag{159}$$

giving no reachable LHS's. From (158).2

$$4\alpha_1\mathtt{c}\underline{\mathtt{a}}\alpha_3\begin{cases}\overset{\alpha_3=\mathtt{a}}{\longleftarrow}\ 5\alpha_1\mathtt{ca}\underline{\mathtt{d}}\\ \overset{\alpha_3=\mathtt{c}}{\longleftarrow}\begin{cases}2\alpha_1\mathtt{ca}\underline{\mathtt{b}}\\ 3\alpha_1\mathtt{ca}\underline{\mathtt{b}}\end{cases}\\ \overset{\alpha_3=\mathtt{d}}{\longleftarrow}\ 1\alpha_1\mathtt{ca}\underline{\mathtt{b}}\\ \dots\end{cases}\tag{160}$$

The corresponding RHS's are easily found to be given by

$$\begin{array}{l}3\mathtt{f}_1(\alpha_1)\mathtt{bc}\underline{\mathtt{a}}\rightarrow 4\mathtt{f}_1(\alpha_1)\mathtt{bcc}\_\\ 3\mathtt{f}_1(\alpha_1)\mathtt{bc}\underline{\mathtt{c}}\rightarrow 2\mathtt{f}_1(\alpha_1)\mathtt{bab}\_\\ 3\mathtt{f}_1(\alpha_1)\mathtt{bc}\underline{\mathtt{d}}\rightarrow 2\mathtt{f}_1(\alpha_1)\mathtt{bdb}\_\end{array}\tag{161}$$

From (158).3

$$2\alpha_1\mathtt{c}\underline{\mathtt{e}}\alpha_3\begin{cases}\overset{\alpha_3=\mathtt{a}}{\longleftarrow}\ 3\alpha_1\mathtt{ce}\underline{\mathtt{c}}\\ \overset{\alpha_3=\mathtt{d}}{\longleftarrow}\ 1\alpha_1\mathtt{ce}\underline{\mathtt{a}}\\ \overset{\alpha_3=\mathtt{e}}{\longleftarrow}\ 5\alpha_1\mathtt{ce}\underline{\mathtt{a}}\end{cases}\tag{162}$$

and the corresponding RHS's are obtained from

$$\begin{array}{l}1\mathtt{f}_1(\alpha_1)\mathtt{bd}\underline{\mathtt{a}}\rightarrow 2\mathtt{f}_1(\alpha_1)\mathtt{bab}\_\\ 1\mathtt{f}_1(\alpha_1)\mathtt{bd}\underline{\mathtt{d}}\rightarrow 2\mathtt{f}_1(\alpha_1)\mathtt{bdb}\_\\ 1\mathtt{f}_1(\alpha_1)\mathtt{bd}\underline{\mathtt{e}}\rightarrow 2\mathtt{f}_1(\alpha_1)\mathtt{bdb}\_\end{array}\tag{163}$$

Now let $\mathtt{C}\alpha$ be the LHS of an AIRR of type $\mathtt{P}$. For each α that makes this true we have say $\mathtt{C}\alpha \leftarrow \mathtt{D}$ of type + demonstrating the reachability of $\mathtt{B}\alpha$ ensuring that $\mathtt{B}\alpha$ is the LHS of a member of IRR($\mathtt{n}+\mathtt{1}$). If the IRR is also of type LR or RL the process can be repeated getting a member of IRR($\mathtt{n}+\mathtt{2}$). This is in outline as follows: suppose there is another AIRR of type $\mathtt{P}$ with $\mathtt{D}$ as LHS. Then there is say $\mathtt{D}\alpha_1 \leftarrow \mathtt{E}$ of type + and $\mathtt{B}\alpha\alpha_1$ is reachable and so is the LHS of a member of IRR($\mathtt{n}+\mathtt{2}$). This process could be never ending, but in practice, because the AIRR branches of type + are involved at each stage, the pointer has to return to the same end of the string, as $\mathtt{n}$ increases it seems less likely that the pointer could traverse the entire length of the string in each case. No doubt there are interesting cases when this happens, but most likely irreducible such rules will have the string usually truncated. A new approach to generating the IRR will be tried based on this idea.

Start by obtaining the IRR(2) as above. Now suppose that IRR($\mathtt{n}$) has been obtained for some $\mathtt{n}\geq 2$. For $\mathtt{X}\in$ IRR($\mathtt{n}$) represent $\mathtt{X}$ as $\mathtt{A}\leftarrow\mathtt{B}\leftarrow\mathtt{C}$. If $\mathtt{X}$ has type LR or RL (pointer at opposite ends in $\mathtt{A}$ and $\mathtt{B}$) add α to the opposite end from the pointer in $\mathtt{B}$ (where α can represent any symbol used by the TM) and add α to $\mathtt{A}$ and $\mathtt{C}$ to match to obtain $\alpha\mathtt{X}$. start the backward search algorithm from $\alpha\mathtt{C}$ searching for branches of type +. Each one must be found to ensure all α for which $\alpha\mathtt{B}$ is reachable have been obtained. For each of these branches the result must be expressed in irreducible form i.e. eliminating any symbols in the string irrelevant to the result. The result of this will be mapping from each + branch of an AIRR of type $\mathtt{P}$ and α to another AIRR. If this is of type $\mathtt{P}$ the process continues. For each such α, a member of IRR($\mathtt{n}+\mathtt{1}$) exists generated by $\alpha\mathtt{B}$.

Finally from (158).4

$$\mathtt{5dc\underline{b}}\alpha_3 \overset{\alpha_3=\mathtt{c}}{\longleftarrow} \begin{cases} \mathtt{3dcb\underline{d}} \\ \mathtt{4dcb\underline{d}} \end{cases} \tag{164}$$

and $\mathtt{1abd\underline{c}} \to \mathtt{4caac\_}$, so the EIRR for $\mathtt{B} = \mathtt{2}\underline{\alpha_1}\mathtt{a}\alpha_2\alpha_3$ are as follows with the number of cases for each formula indicated on the right.

α_1	α_2	α_3	$\mathtt{C} \to \mathtt{A}$	No. of EIRR $\mathtt{B} \to \mathtt{A}$
$\{\mathtt{a},\mathtt{c},\mathtt{d}\}$	$\mathtt{b}$	$\mathtt{a}$	$\mathtt{5}\alpha_1\mathtt{ca\underline{d}} \to \mathtt{4f}_1(\alpha_1)\mathtt{bcc\_}$	3
$\{\mathtt{a},\mathtt{c},\mathtt{d}\}$	$\mathtt{b}$	$\mathtt{c}$	$\left\{\begin{matrix}\mathtt{2}\\\mathtt{3}\end{matrix}\right\}\alpha_1\mathtt{ca\underline{b}} \to \mathtt{2f}_1(\alpha_1)\mathtt{bab\_}$	3
$\{\mathtt{a},\mathtt{c},\mathtt{d}\}$	$\mathtt{b}$	$\mathtt{d}$	$\mathtt{1}\alpha_1\mathtt{ca\underline{b}} \to \mathtt{2f}_1(\alpha_1)\mathtt{bdb\_}$	3
$\{\mathtt{a},\mathtt{c},\mathtt{d}\}$	$\mathtt{c}$	$\mathtt{a}$	$\mathtt{3}\alpha_1\mathtt{ce\underline{c}} \to \mathtt{2f}_1(\alpha_1)\mathtt{bab\_}$	3
$\{\mathtt{a},\mathtt{c},\mathtt{d}\}$	$\mathtt{c}$	$\mathtt{d}$	$\mathtt{1}\alpha_1\mathtt{ce\underline{a}} \to \mathtt{2f}_1(\alpha_1)\mathtt{bdb\_}$	3
$\{\mathtt{a},\mathtt{c},\mathtt{d}\}$	$\mathtt{c}$	$\mathtt{e}$	$\mathtt{5}\alpha_1\mathtt{ce\underline{a}} \to \mathtt{2f}_1(\alpha_1)\mathtt{bdb\_}$	3
$\mathtt{d}$	$\mathtt{e}$	$\mathtt{c}$	$\left\{\begin{matrix}\mathtt{3}\\\mathtt{4}\end{matrix}\right\}\mathtt{dcb\underline{d}} \to \mathtt{4caac\_}$	1

(165)

4.4 $\mathtt{B} = \mathtt{2}\underline{\alpha_1}\mathtt{a}\alpha_2\alpha_3\alpha_4$

While carrying out the next stage of the calculations to derive the EIRR of length 5 from the EIRR listed in (165) the following ARR were needed. It is interesting to note that each of these except the first is reducible to a single reverse rule on page 2. This suggests that a large proportion of the ARR used in the derivations of all the EIRR are of this simple type.

$$\mathtt{5}\alpha_1\mathtt{ca\underline{d}}\alpha_4 \begin{cases} \overset{\alpha_4=\mathtt{c}}{\longleftarrow} \left\{\begin{matrix}\mathtt{3}\\\mathtt{4}\end{matrix}\right\}\alpha_1\mathtt{cad\underline{d}} \\ \ldots \end{cases} \tag{166}$$

$$\mathtt{2}\alpha_1\mathtt{ca\underline{b}}\alpha_4 \begin{cases} \overset{\alpha_4=\mathtt{a}}{\longleftarrow} \mathtt{3}\alpha_1\mathtt{cab\underline{c}} \\ \overset{\alpha_4=\mathtt{d}}{\longleftarrow} \mathtt{1}\alpha_1\mathtt{cab\underline{a}} \\ \overset{\alpha_4=\mathtt{e}}{\longleftarrow} \mathtt{5}\alpha_1\mathtt{cab\underline{a}} \end{cases} \tag{167}$$

$$3\alpha_1\texttt{ca}\underline{\texttt{b}}\alpha_4\left\{\begin{array}{l}\stackrel{\alpha_4=\texttt{a}}{\longleftarrow}1\alpha_1\texttt{cab}\underline{\texttt{c}}\\\stackrel{\alpha_4=\texttt{b}}{\longleftarrow}4\alpha_1\texttt{cab}\underline{\texttt{a}}\\\stackrel{\alpha_4=\texttt{c}}{\longleftarrow}2\alpha_1\texttt{cab}\underline{\texttt{e}}\\\stackrel{\alpha_4=\texttt{e}}{\longleftarrow}5\alpha_1\texttt{cab}\underline{\texttt{b}}\end{array}\right.\tag{168}$$

$$1\alpha_1\texttt{ca}\underline{\texttt{b}}\alpha_4\leftarrow\emptyset\tag{169}$$

$$3\alpha_1\texttt{ce}\underline{\texttt{c}}\alpha_4\left\{\begin{array}{l}\stackrel{\alpha_4=\texttt{a}}{\longleftarrow}1\alpha_1\texttt{cec}\underline{\texttt{c}}\\\stackrel{\alpha_4=\texttt{b}}{\longleftarrow}4\alpha_1\texttt{cec}\underline{\texttt{a}}\\\stackrel{\alpha_4=\texttt{c}}{\longleftarrow}2\alpha_1\texttt{cec}\underline{\texttt{e}}\\\stackrel{\alpha_4=\texttt{e}}{\longleftarrow}5\alpha_1\texttt{cec}\underline{\texttt{b}}\end{array}\right.\tag{170}$$

$$1\alpha_1\texttt{ce}\underline{\texttt{a}}\alpha_4\leftarrow\emptyset\tag{171}$$

$$5\alpha_1\texttt{ce}\underline{\texttt{a}}\alpha_4\stackrel{\alpha_4=\texttt{c}}{\longleftarrow}\left\{\begin{array}{c}3\\4\end{array}\right\}\alpha_1\texttt{cea}\underline{\texttt{d}}\tag{172}$$

$$3\texttt{dcb}\underline{\texttt{d}}\alpha_4\left\{\begin{array}{l}\stackrel{\alpha_4=\texttt{a}}{\longleftarrow}1\texttt{dcbd}\underline{\texttt{c}}\\\stackrel{\alpha_4=\texttt{b}}{\longleftarrow}4\texttt{dcbd}\underline{\texttt{a}}\\\stackrel{\alpha_4=\texttt{c}}{\longleftarrow}2\texttt{dcbd}\underline{\texttt{e}}\\\stackrel{\alpha_4=\texttt{e}}{\longleftarrow}5\texttt{dcbd}\underline{\texttt{b}}\end{array}\right.\tag{173}$$

$$4\texttt{dcb}\underline{\texttt{d}}\alpha_4\left\{\begin{array}{l}\stackrel{\alpha_4=\texttt{a}}{\longleftarrow}5\texttt{dcbd}\underline{\texttt{d}}\\\stackrel{\alpha_4=\texttt{c}}{\longleftarrow}\left\{\begin{array}{l}2\texttt{dcbd}\underline{\texttt{b}}\\3\texttt{dcbd}\underline{\texttt{b}}\end{array}\right.\\\stackrel{\alpha_4=\texttt{d}}{\longleftarrow}1\texttt{dcbd}\underline{\texttt{b}}\end{array}\right.\tag{174}$$

The resulting IRR of length 5 for $\mathtt{B} = 2\underline{\alpha_1}\mathtt{a}\alpha_2\alpha_3\alpha_4$ are as described in the following table:

α_1	α_2	α_3	α_4	$\mathtt{C} \to \mathtt{A}$
$\{\mathtt{a},\mathtt{c},\mathtt{d}\}$	b	a	c	$\left\{\begin{matrix}\mathtt{3}\\\mathtt{4}\end{matrix}\right\}\alpha_1\mathtt{cad}\underline{\mathtt{d}} \to \mathtt{3f}_1(\alpha_1)\mathtt{bccc\_}$
$\{\mathtt{a},\mathtt{c},\mathtt{d}\}$	b	c	a	$\left\{\begin{matrix}\mathtt{1}\\\mathtt{3}\end{matrix}\right\}\alpha_1\mathtt{cab}\underline{\mathtt{c}} \to \mathtt{1f}_1(\alpha_1)\mathtt{babc\_}$
$\{\mathtt{a},\mathtt{c},\mathtt{d}\}$	b	c	b	$\mathtt{4}\alpha_1\mathtt{cab}\underline{\mathtt{a}} \to \mathtt{3f}_1(\alpha_1)\mathtt{babc\_}$
$\{\mathtt{a},\mathtt{c},\mathtt{d}\}$	b	c	c	$\mathtt{2}\alpha_1\mathtt{cab}\underline{\mathtt{e}} \to \mathtt{1f}_1(\alpha_1)\mathtt{babd\_}$
$\{\mathtt{a},\mathtt{c},\mathtt{d}\}$	b	c	d	$\mathtt{1}\alpha_1\mathtt{cab}\underline{\mathtt{a}} \to \mathtt{1f}_1(\alpha_1)\mathtt{baba\_}$
a	b	c	e	$\mathtt{5}\alpha_1\mathtt{cab}\left\{\begin{matrix}\mathtt{a}\\\mathtt{b}\end{matrix}\right\} \to \mathtt{5\_ccbcc}$
c	b	c	e	$\mathtt{5}\alpha_1\mathtt{cab}\left\{\begin{matrix}\mathtt{a}\\\mathtt{b}\end{matrix}\right\} \to \mathtt{3\_bcbcc}$
d	b	c	e	$\mathtt{5}\alpha_1\mathtt{cab}\left\{\begin{matrix}\mathtt{a}\\\mathtt{b}\end{matrix}\right\} \to \mathtt{1ababd\_}$
$\{\mathtt{a},\mathtt{c},\mathtt{d}\}$	c	a	a	$\mathtt{1}\alpha_1\mathtt{cec}\underline{\mathtt{c}} \to \mathtt{1f}_1(\alpha_1)\mathtt{bab}\underline{\mathtt{c}}\mathtt{\_}$
$\{\mathtt{a},\mathtt{c},\mathtt{d}\}$	c	a	b	$\mathtt{4}\alpha_1\mathtt{cec}\underline{\mathtt{a}} \to \mathtt{3f}_1(\alpha_1)\mathtt{babc\_}$
$\{\mathtt{a},\mathtt{c},\mathtt{d}\}$	c	a	c	$\mathtt{2}\alpha_1\mathtt{cec}\underline{\mathtt{e}} \to \mathtt{1f}_1(\alpha_1)\mathtt{babd\_}$
a	c	a	e	$\mathtt{5}\alpha_1\mathtt{cab}\left\{\begin{matrix}\mathtt{a}\\\mathtt{b}\end{matrix}\right\} \to \mathtt{5\_ccbcc}$
c	c	a	e	$\mathtt{5}\alpha_1\mathtt{cab}\left\{\begin{matrix}\mathtt{a}\\\mathtt{b}\end{matrix}\right\} \to \mathtt{3\_bcbcc}$
d	c	a	e	$\mathtt{5}\alpha_1\mathtt{cab}\left\{\begin{matrix}\mathtt{a}\\\mathtt{b}\end{matrix}\right\} \to \mathtt{1ababd\_}$
$\{\mathtt{a},\mathtt{c},\mathtt{d}\}$	c	e	c	$\left\{\begin{matrix}\mathtt{3}\\\mathtt{4}\end{matrix}\right\}\alpha_1\mathtt{cea}\underline{\mathtt{d}} \to \mathtt{1f}_1(\alpha_1)\mathtt{bdbd\_}$
d	e	c	a	$\left\{\begin{matrix}\mathtt{1dcbd}\underline{\mathtt{c}}\\\mathtt{5dcbd}\underline{\mathtt{d}}\end{matrix}\right\} \to \mathtt{3cadbc\_}$
d	e	c	b	$\mathtt{4dcbd}\underline{\mathtt{a}} \to \mathtt{4caacb\_}$
d	e	c	c	$\left\{\begin{matrix}\mathtt{2dcbd}\underline{\mathtt{e}}\\\mathtt{2dcbd}\underline{\mathtt{b}}\\\mathtt{3dcbd}\underline{\mathtt{b}}\end{matrix}\right\} \to \mathtt{3caacc\_}$
d	e	c	d	$\mathtt{1dcbd}\underline{\mathtt{b}} \to \mathtt{2caadb\_}$
d	e	c	e	$\mathtt{5dcbd}\underline{\mathtt{b}} \to \mathtt{5caaca\_}$

(175)

****

By repeatedly using the increment function, any AIRR can be developed starting with the AIRR truncated to length 3. This technique was found to be useful in the following derivation in the next section. Also the process of finding the appropriate AIRR C for each RHS of B in Theorem 2.2 may be a lengthy process. One way I found to be very useful in a complex case was to first truncate Z the RHS of B to length 3, and do the single step to find the AIRR X generated by this. Then apply the method indicated in Theorem 2.1 to X in the place of B to find the AIRR generated by Z truncated to length 4. This involves bringing back one symbol from Z. This process is repeated until either the complete Z has been obtained or until the AIRR generated by the truncated Z is of type + or $\emptyset$. The same technique can be applied to generating the AIRR involved in Theorem 2.1 itself because theorems 2.1 and 2.2 do not differ in the form of the RHS's to be derived. Thus a recursive function may be used to carry this out.

Repeat the process of getting from A to D to get from say D to E. This will involve another set of AIRR say F similar to C. Characterise the relationship between the sets of of AIRR generically called C and F.

The motivation for introducing AIRR is as a more compact way of describing the IRR, because the derivation of the IRR as outlined above often involves repeated use of the same backward searches which can be defined as AIRR. What is desired is a procedure for finitely characterising the AIRR that are needed for deriving the IRR for a TM.

****

$$\begin{array}{l} 2\underline{\beta}\gamma\mathtt{be}\ldots \to \mathtt{3f_2}(\beta)\mathtt{bcb}\ldots \\ 2\underline{\beta}\gamma\mathtt{ba}\ldots \to \mathtt{4f_2}(\beta)\mathtt{bcc}\ldots \end{array} \quad \text{for } \beta \in \{\mathtt{a},\mathtt{c},\mathtt{d}\} \text{ and } \gamma \in \{\mathtt{d},\mathtt{e}\} \tag{176}$$

where

x	f_2(x)
a	c
c	d
d	a

(177)

$$5\underline{\delta}\mathtt{eb}\epsilon\ldots \to \begin{cases} \overset{\epsilon=\mathtt{e}}{\to} \mathtt{3abcb}\ldots \\ \overset{\epsilon=\mathtt{a}}{\to} \mathtt{4abcc}\ldots \end{cases} \tag{178}$$

$$3\underline{\mathtt{e}}\mathtt{eb}\epsilon\ldots \to \begin{cases} \overset{\epsilon=\mathtt{e}}{\to} \mathtt{3bbcb}\ldots \\ \overset{\epsilon=\mathtt{a}}{\to} \mathtt{4bbcc}\ldots \end{cases} \tag{179}$$

$$4\underline{\mathtt{c}}\mathtt{eb}\epsilon\ldots \to \begin{cases} \overset{\epsilon=\mathtt{e}}{\to} \mathtt{3cbcb}\ldots \\ \overset{\epsilon=\mathtt{a}}{\to} \mathtt{4cbcc}\ldots \end{cases} . \tag{180}$$

****

Table 1: List of some outline IRRs and their derived outline IRR with length increased by 1

IRR(n)			IRR(n + 1)		
Origin	LHS	RHS	Origin	LHS	RHS
1da...	1bc...	4_ca...	2dd...	1ab...	3_bc...
			2ad...	1cb...	1ab...
			2cd...	1db...	5_cc...
1da...	2bc...	4_ca...	2dd...	2ab...	3_bc...
			2ad...	2cb...	1ab...
			2cd...	2db...	5_cc...
1dc...	1bd...	3_ec...	2dd...	1ab...	3ca...
			2ad...	1cb...	2_ae...
			2cd...	1db...	5_ce...
1dc...	2bd...	3_ec...	2dd...	2ab...	3ca...
			2ad...	2cb...	2_ae...
			2cd...	2db...	5_ce...
1dd...	1ba...	4_ca...	2dd...	1ab...	3_bc...
			2ad...	1cb...	1ab...
			2cd...	1db...	5_cc...
1dd...	1ba...	4_cb...	2dd...	1ab...	3_bc...
			2ad...	1cb...	3ab...
			2cd...	1db...	5_cc...
1dd...	2ba...	4_cb...	2dd...	2ab...	3_bc...
			2ad...	2cb...	3ab...
			2cd...	2db...	5_cc...
1ea...	1bc...	4_ca...	2de...	1ab...	3_bc...
			2ae...	1cb...	1ab...
			2ce...	1db...	5_cc...
1ea...	2bc...	4_ca...	2de...	2ab...	3_bc...
			2ae...	2cb...	1ab...
			2ce...	2db...	5_cc...
1ec...	1bd...	3_ec...	2de...	1ab...	3ca...
			2ae...	1cb...	2_ae...
			2ce...	1db...	5_ce...
1ec...	2bd...	3_ec...	2de...	2ab...	3ca...
			2ae...	2cb...	2_ae...
			2ce...	2db...	5_ce...
1ed...	1ba...	4_ca...	2de...	1ab...	3_bc...
			2ae...	1cb...	1ab...
			2ce...	1db...	5_cc...
1ed...	1ba...	4_cb...	2de...	1ab...	3_bc...

			2ae...	1cb...	3ab...
			2ce...	1db...	5_cc...
1ed...	2ba...	4_cb...	2de...	2ab...	3_bc...
			2ae...	2cb...	3ab...
			2ce...	2db...	5_cc...
2ad...	1cb...	2_ae...	1da...	1bc...	4_ca...
			1ea...	1bc...	4_ca...
2ad...	2cb...	2_ae...	1da...	2bc...	4_ca...
			1ea...	2bc...	4_ca...
2ae...	1cb...	2_ae...	1da...	1bc...	4_ca...
			1ea...	1bc...	4_ca...
2ae...	2cb...	2_ae...	1da...	2bc...	4_ca...
			1ea...	2bc...	4_ca...
2cd...	1db...	5_cc...	1dc...	1bd...	3_ec...
			1ec...	1bd...	3_ec...
2cd...	1db...	5_ce...	1dc...	1bd...	3_ec...
			1ec...	1bd...	3_ec...
2cd...	2db...	5_cc...	1dc...	2bd...	3_ec...
			1ec...	2bd...	3_ec...
2cd...	2db...	5_ce...	1dc...	2bd...	3_ec...
			1ec...	2bd...	3_ec...
2ce...	1db...	5_cc...	1dc...	1bd...	3_ec...
			1ec...	1bd...	3_ec...
2ce...	1db...	5_ce...	1dc...	1bd...	3_ec...
			1ec...	1bd...	3_ec...
2ce...	2db...	5_cc...	1dc...	2bd...	3_ec...
			1ec...	2bd...	3_ec...
2ce...	2db...	5_ce...	1dc...	2bd...	3_ec...
			1ec...	2bd...	3_ec...
2dd...	1ab...	3_bc...	1dd...	1ba...	4_cb...
			1ed...	1ba...	4_cb...
2dd...	2ab...	3_bc...	1dd...	2ba...	4_cb...
			1ed...	2ba...	4_cb...
2de...	1ab...	3_bc...	1dd...	1ba...	4_cb...
			1ed...	1ba...	4_cb...
2de...	2ab...	3_bc...	1dd...	2ba...	4_cb...
			1ed...	2ba...	4_cb...
3ab...	4cb...	2_ae...	5ca...	4ac...	5_cc...
			5ea...	4ac...	5_cc...
			3ea...	4bc...	4_ca...
			4ca...	4cc...	3_bc...
3ab...	4cb...	3_bc...	5ca...	4ac...	3cb...

			5ea...	4ac...	3cb...
			3ea...	4bc...	4_cb...
			4ca...	4cc...	2_ab...
3ac...	3cc...	2_ab...	5ca...	3ac...	3db...
			5ea...	3ac...	3db...
			3ea...	3bc...	4_ca...
			4ca...	3cc...	3ab...
3ac...	3cc...	3_bc...	5ca...	3ac...	3cb...
			5ea	3ac...	3cb...
			3ea...	3bc...	4_cb...
			4ca...	3cc...	2_ab...
3ac...	4cc...	2_ab...	5ca...	4ac...	3db...
			5ea...	4ac...	3db...
			3ea...	4bc...	4_ca...
			4ca...	4cc...	3ab...
3ae...	5ca...	2_ab...	5ca...	5ac...	3db...
			5ea...	5ac...	3db...
			3ea...	5bc...	4_ca...
			4ca...	5cc...	3ab...
3ea...	3bc...	4_ca...	5ce...	3ab...	3_bc...
			5ee...	3ab...	3_bc...
			3ee...	3bb...	4bc...
			4ce...	3cb...	1ab...
3ea...	3bc...	4_cb...	5ce...	3ab...	3_bc...
			5ee...	3ab...	3_bc...
			3ee...	3bb...	2ba...
			4ce...	3cb...	3ab...
3ea...	4bc...	4_ca...	5ce...	4ab...	3_bc...
			5ee...	4ab...	3_bc...
			3ee...	4bb...	4bc...
			4ce...	4cb...	1ab...
3ea...	4bc...	4_cb...	5ce...	4ab...	3_bc...
			5ee...	4ab...	3_bc...
			3ee...	4bb...	2ba...
			4ce...	4cb...	3ab...
3ea...	5bc...	4_ca...	5ce...	5ab...	3_bc...
			5ee...	5ab...	3_bc...
			3ee...	5bb...	4bc...
			4ce...	5cb...	1ab...
3ee...	3bb...	4_ce...	5ce...	3ab...	3_bc...
			5ee...	3ab...	3_bc...
			3ee...	3bb...	3bc...

			4ce...	3cb...	3_bc...
4bb...	4bb...	4_ce...	4bb...	4bb...	3bc...
			3ab...	4cb...	3_bc...
4bc...	3bc...	4_ca...	4bb...	3bb...	4bc...
			3ab...	3cb...	1ab...
4bc...	3bc...	4_cb...	4bb...	3bb...	2ba...
			3ab...	3cb...	3ab...
4bc...	4bc...	4_ca...	4bb...	4bb...	4bc...
			3ab...	4cb...	1ab...
4bc...	4bc...	4_cb...	4bb...	4bb...	2ba...
			3ab...	4cb...	3ab...
4be...	4ba...	4_ce...	4bb...	4bb...	3bc...
			3ab...	4cb...	3_bc...
4be...	5ba...	4_cb...	4bb...	5bb...	2ba...
			3ab...	5cb...	3ab...
4ca...	3cc...	2_ab...	4bc...	3bc...	4_ca...
			3ac...	3cc...	3ab...
4ca...	3cc...	2_ac...	4bc...	3bc...	4_ca...
			3ac...	3cc...	1ab...
4ca...	4cc...	2_ab...	4bc...	4bc...	4_ca...
			3ac...	4cc...	3ab...
4ca...	4cc...	3_bc...	4bc...	4bc...	4_cb...
			3ac...	4cc...	2_ab...
4ce...	3cb...	2_ae...	4bc...	3bc...	4_ca...
			3ac...	3cc...	3_bc...
4ce...	3cb...	3_bc...	4bc...	3bc...	4_cb...
			3ac...	3cc...	2_ab...
4ea...	4aa...	2_ec...	4be...	4ba...	4_ce...
			3ae...	4ca...	1db...
4ec...	4aa...	2_ec...	4be...	4ba...	4_ce...
			3ae...	4ca...	1db...
4ee...	4aa...	2_ec...	4be...	4ba...	4_ce...
			3ae...	4ca...	1db...
5ca...	4ac...	5_cc...	4ec...	4aa...	2_ec...
5ce...	3ab...	3_bc...	4ec...	3aa...	3cb...
5ce...	4ab...	3_bc...	4ec...	4aa...	3cb...
5ce...	5ab...	3_bc...	4ec...	5aa...	3cb...
5ea...	4ac...	5_cc...	4ee...	4aa...	2_ec...
5ee...	3ab...	3_bc...	4ee...	3aa...	3cb...
5ee...	4ab...	3_bc...	4ee...	4aa...	3cb...
5ee...	5ab...	3_bc...	4ee...	5aa...	3cb...

Table 2: List of some outline IRRs and their derived outline IRR with length increased by 1

IRR(n)			IRR(n + 1)		
Origin	LHS	RHS	Origin	LHS	RHS
2...a$\underline{\text{b}}$	3...bc	2...ab_	3...b$\underline{\text{c}}$	3...ca	1...bc_
			1...b$\underline{\text{a}}$	3...cd	1...ba_
2...a$\underline{\text{b}}$	3...bc	3...bc_	3...b$\underline{\text{c}}$	3...ca	4...cc_
			1...b$\underline{\text{a}}$	3...cd	2...db_
			5...b$\underline{\text{a}}$	3...ce	3...cb_
2...d$\underline{\text{b}}$	5...cc	1...bd_	3...b$\underline{\text{c}}$	5...ca	2...ab_
			1...b$\underline{\text{a}}$	5...cd	2...db_
			5...b$\underline{\text{a}}$	5...ce	2...db_
2...b$\underline{\text{e}}$	4...cc	1...bd_	3...e$\underline{\text{c}}$	4...ca	2...ab_
			1...e$\underline{\text{a}}$	4...cd	2...db_
			5...c$\underline{\text{a}}$	4...ce	2...db_
2...b$\underline{\text{e}}$	4...cc	2...ab_	3...e$\underline{\text{c}}$	4...ca	1...bc_
			1...e$\underline{\text{a}}$	4...cd	1...ba_
2...b$\underline{\text{e}}$	4...cc	2...db_	3...e$\underline{\text{c}}$	4...ca	1...bc_
			1...e$\underline{\text{a}}$	4...cd	1...ba_
2...c$\underline{\text{e}}$	2...ac	1...bd_	3...e$\underline{\text{c}}$	2...ca	2...ab_
			1...e$\underline{\text{a}}$	2...cd	2...db_
			5...e$\underline{\text{a}}$	2...ce	2...db_
2...d$\underline{\text{e}}$	5...cc	1...bd_	3...e$\underline{\text{c}}$	5...ca	2...ab_
			1...e$\underline{\text{a}}$	5...cd	2...db_
			5...e$\underline{\text{a}}$	5...ce	2...db_
3...a$\underline{\text{b}}$	3...bc	2...ab_	1...b$\underline{\text{c}}$	3...ca	1...bc_
			4...b$\underline{\text{a}}$	3...cb	3...bc_
			2...b$\underline{\text{e}}$	3...cc	1...bd_
3...a$\underline{\text{b}}$	3...bc	3...bc_	1...b$\underline{\text{c}}$	3...ca	4...cc_
			4...b$\underline{\text{a}}$	3...cb	2...ab_
			2...b$\underline{\text{e}}$	3...cc	2...ab_
			5...b$\underline{\text{b}}$	3...ce	3...cb_
3...d$\underline{\text{b}}$	5...cc	1...bd_	1...b$\underline{\text{c}}$	5...ca	2...ab_
			5...b$\underline{\text{b}}$	5...ce	2...db_
3...b$\underline{\text{c}}$	4...ca	1...bc_	1...c$\underline{\text{c}}$	4...aa	2...db_
			4...c$\underline{\text{a}}$	4...ab	2...db_
			5...c$\underline{\text{b}}$	4...ae	2...cb_
3...b$\underline{\text{c}}$	4...ca	4...ac_	1...c$\underline{\text{c}}$	4...aa	3...bc_
			4...c$\underline{\text{a}}$	4...ab	4...cb_
			2...c$\underline{\text{e}}$	4...ac	3...cc_
			5...c$\underline{\text{b}}$	4...ae	5...ca_

3...b$\underline{c}$	4...ca	4...cc_	1...c$\underline{c}$	4...aa	3...bc_
			4...c$\underline{a}$	4...ab	4...cb_
			2...c$\underline{e}$	4...ac	3...cc_
			5...c$\underline{b}$	4...ae	5...ca_
3...e$\underline{c}$	3...ca	4...cc_	1...c$\underline{c}$	3...aa	3...bc_
			4...c$\underline{c}$	3...ab	4...cb_
			2...c$\underline{e}$	3...ac	3...cc_
			5...c$\underline{b}$	3...ae	5...ca_
3...a$\underline{d}$	2...ec	1...bd_	1...d$\underline{c}$	2...ca	2...ab_
			5...d$\underline{b}$	2...ce	2...db_
3...b$\underline{d}$	3...ec	3...aa_	1...d$\underline{c}$	3...ca	4...ac_
			4...d$\underline{a}$	3...cb	3...bc_
			2...d$\underline{e}$	3...cc	2...db_
			5...d$\underline{b}$	3...ce	3...ab_
3...b$\underline{d}$	3...ec	4...cc_	1...d$\underline{c}$	3...ca	3...bc_
			4...d$\underline{a}$	3...cb	4...cb_
			2...d$\underline{e}$	3...cc	3...cc_
			5...d$\underline{b}$	3...ce	5...ca_
3...d$\underline{d}$	4...ac	1...bd_	1...d$\underline{c}$	4...ca	2...ab_
			5...d$\underline{b}$	4...ce	2...db_
3...d$\underline{d}$	4...ac	3...cc_	1...d$\underline{c}$	4...ca	4...cc_
			4...d$\underline{a}$	4...cb	2...ab_
			2...d$\underline{e}$	4...cc	2...ab_
			5...d$\underline{b}$	4...ce	3...cb_
4...b$\underline{a}$	4...cb	2...ab_	5...a$\underline{d}$	4...ba	1...bc_
			2...a$\underline{b}$	4...bc	1...bd_
			3...a$\underline{b}$	4...bc	1...bd_
			1...a$\underline{b}$	4...bd	1...ba_
4...b$\underline{a}$	4...cb	3...bc_	5...a$\underline{d}$	4...ba	4...cc_
			2...a$\underline{b}$	4...bc	2...ab_
			3...a$\underline{b}$	4...bc	2...ab_
			1...a$\underline{b}$	4...bd	2...db_
4...c$\underline{a}$	2...ab	3...bc_	5...a$\underline{d}$	2...ba	4...cc_
			2...a$\underline{b}$	2...bc	2...ab_
			3...a$\underline{b}$	2...bc	2...ab_
			1...a$\underline{b}$	2...bd	2...db_
4...d$\underline{a}$	5...cb	3...bc_	5...a$\underline{d}$	5...ba	4...cc_
			2...a$\underline{b}$	5...bc	2...ab_
			3...a$\underline{b}$	5...bc	2...ab_
			1...a$\underline{b}$	5...bd	2...db_
4...a$\underline{d}$	2...ec	1...bd_	5...d$\underline{d}$	2...ca	2...ab_

			1...d$\underline{b}$	2...cd	2...db_
4...b$\underline{d}$	3...ec	3...aa_	5...d$\underline{d}$	3...ca	4...ac_
			2...d$\underline{b}$	3...cc	2...db_
			3...d$\underline{b}$	3...cc	2...db_
			1...d$\underline{b}$	3...cd	1...bd_
4...b$\underline{d}$	3...ec	4...cc_	5...d$\underline{d}$	3...ca	3...bc_
			2...d$\underline{b}$	3...cc	3...cc_
			3...d$\underline{b}$	3...cc	3...cc_
			1...d$\underline{b}$	3...cd	2...db_
4...d$\underline{d}$	4...ac	1...bd_	5...d$\underline{d}$	4...ca	2...ab_
			1...d$\underline{b}$	4...cd	2...db_
4...d$\underline{d}$	4...ac	3...cc_	5...d$\underline{d}$	4...ca	4...cc_
			2...d$\underline{b}$	4...cc	2...ab_
			3...d$\underline{b}$	4...cc	2...ab_
			1...d$\underline{b}$	4...cd	2...db_
5...b$\underline{a}$	4...ce	3...cb_	3...a$\underline{d}$	4...ec	1...bc_
			4...a$\underline{d}$	4...ec	1...bc_
5...e$\underline{a}$	3...ce	3...cb_	3...a$\underline{d}$	3...ec	1...bc_
			4...a$\underline{d}$	3...ec	1...bc_
5...b$\underline{b}$	4...ce	3...cb_	3...b$\underline{d}$	4...ec	1...bc_
			4...b$\underline{d}$	4...ec	1...bc_
5...a$\underline{d}$	3...ba	3...bc_	3...d$\underline{d}$	3...ac	2...ab_
			4...d$\underline{d}$	3...ac	2...ab_
5...a$\underline{d}$	3...ba	4...cc_	3...d$\underline{d}$	3...ac	3...cc_
			4...d$\underline{d}$	3...ac	3...cc_

****

Origin	LHS of IRR(4)	RHS's	
$1\underline{d}ca\ldots$	$2bab\ldots$	$4\_cbc\ldots$	
$1\underline{e}ca\ldots$	$2bab\ldots$	$4\_cbc\ldots$	
$4\underline{b}cc\ldots$	$2bab\ldots$	$4\_cbc\ldots$	
$4\underline{b}cb\ldots$	$2bab\ldots$	$4\_cbc\ldots$	
$1\underline{d}da\ldots$	$2bab\ldots$	$4\_cbc\ldots$	
$1\underline{e}da\ldots$	$2bab\ldots$	$4\_cbc\ldots$	
$3\underline{a}cc\ldots$	$2cab\ldots$	$2\_abc\ldots$	
$3\underline{a}cb\ldots$	$2cab\ldots$	$2\_abc\ldots$	
$1\underline{d}aa\ldots$	$2bdb\ldots$	$3\_ecc\ldots$	
$4\underline{b}cd\ldots$	$2bdb\ldots$	$3\_ecc\ldots$	
$1\underline{d}ab\ldots$	$2bdb\ldots$	$3\_ecc\ldots$	
$1\underline{e}aa\ldots$	$2bdb\ldots$	$3\_ecc\ldots$	
$3\underline{e}ad\ldots$	$2bdb\ldots$	$3\_ecc\ldots$	(181)
$1\underline{e}ab\ldots$	$2bdb\ldots$	$3\_ecc\ldots$	
$4\underline{c}ad\ldots$	$2bdb\ldots$	$3\_ecc\ldots$	
$4\underline{e}cc\ldots$	$2adb\ldots$	$2\_ecc\ldots$	
$4\underline{e}ec\ldots$	$2adb\ldots$	$2\_ecc\ldots$	
$5\underline{c}ad\ldots$	$2adb\ldots$	$2\_ecc\ldots$	
$5\underline{e}ad\ldots$	$2adb\ldots$	$2\_ecc\ldots$	
$3\underline{a}cd\ldots$	$2cdb\ldots$	$5\_cec\ldots$	
$3\ldots ee\underline{c}$	$4\ldots caa$	$\{3\ldots abc\_, 2\ldots bdb\_\}$	
$1\ldots ee\underline{a}$	$4\ldots cad$	$\{2\ldots cdb\_, 2\ldots bcb\_, 2\ldots adb\_\}$	
$5\ldots ee\underline{a}$	$4\ldots cae$	$\{5\ldots cca\_, 2\ldots bcb\_, 5\ldots aca\_\}$	
$3\ldots bd\underline{d}$	$4\ldots cbc$	$\{1\ldots abd\_, 2\ldots bab\_\}$	
$4\ldots bd\underline{d}$	$4\ldots cbc$	$\{1\ldots abd\_, 2\ldots bab\_\}$	
$4\underline{b}eb\ldots$	$5bca\ldots$	$4\_cab\ldots$	
$3\underline{a}eb\ldots$	$5cca\ldots$	$4abc\ldots$	

Proceeding as before with arguments like (**??**) applied to each line of (**??**), now starting from (181) the + branches are followed up first. The first line of (**??**) gave the results

$$\begin{array}{l} 2\underline{d}dc\ldots \to 2aba\ldots \to 3\_bcb\ldots \\ 2\underline{a}dc\ldots \to 2cba\ldots \to 2aba\ldots \\ 2\underline{c}dc\ldots \to 2dba\ldots \to 5\_ccb\ldots \end{array} \qquad (182)$$

Of these, the first and third are contained in (**??**) lines 11 and 15 respectively so do not appear to need to be mentioned again. The remaining one was put in (183). Likewise results of contained in (**??**)-(**??**) will not be listed separately.

All that remain will be included in (183).

Origin	LHS of IRR(5)	RHS's
4bca...	2bbd...	
3aca...	2cbd...	
1...ecc	4...aaa	
1...ddc	4...bca	
1...ddb	4...bcd	
5cac...	2abc...	
5eac...	2abc...	
3eac...	2cbc...	
4cac...	2cbc...	
5cea...	2aab...	
5eea...	2aab...	
3eea...	2bab...	
4cea...	2cab...	

(183)

Origin	LHS of IRR(5)	RHS's
2adc...	2cba...	2aba...
2aec...	2cba...	2aba...
4bbc...	2bba...	1bab...
3abc...	2cba...	2aba...
2add...	2cba...	2aba...
2aed...	2cba...	2aba...
5cac...	2aca...	2dba...
5eac...	2aca...	2dba...
3eac...	2bca...	4_dab...
4cac...	2cca...	2aba...
2dda...	2abd...	2cad...
2ada...	2cbd...	2_aec...
2cda...	2dbd...	5_cec...

(184)

****

References

[1]

[2] John Nixon Reverse engineering Turing Machines and the Collatz Conjecture

[3] The updated version in D of the computer program for analysis of Turing Machines