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Towards a Theory of Analytic Functions

Section 7 is still rather a mess but it is getting clearer. Sections 10 and following are just bits and pieces that are mostly probably not needed.

Abstract

Multivalued analytic functions (or relations) are defined as mappings of the Riemann Sphere to itself that satisfy the Cauchy-Riemann equations, and are not constrained by artificial boundaries or constraints on values. They are believed to be determined uniquely by their behaviours at all their singular and inversion points, which is a generalisation of a result of the previous study of the algebraic case. The behaviour at these points is determined by simple equations that only make sense in the context of multivalued functions and can describe behaviour near essential singular points as well as simple poles and branch points associated with algebraic functions. Many examples are discussed. It is suggested though not yet proved that the set of analytic functions forms a large algebraic structure that is closed under the operation of taking limits in addition to the operations that give closure to the set of algebraic functions.

The approach will be intuitive and non-technical showing how to handle multi-valued functions in calculations and the topological properties of the surfaces representing them.

Mathematics Subject Classification:
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1 Introduction

This document is a work in progress. As such it is incomplete and still has errors and omissions. When brought to a state where I cannot easily find any improvements it will form my next document on Complex analysis. Now it looks as if there are going to be so many ideas that I can't just finish it as a paper, it is instead a sort of discussion document.

A strange feature of this study is that as it develops sections get expanded with different material so the section headings get out of date, and it is not easy to get the ideas in the most sensible order and keep it that way. The structure is still obviously not right. Thus there are many places where there are forward references. Comments are welcome. Please send them to john.h.nixon1@gmail.com (see also https://www.bluesky-home.co.uk for my other papers and ideas)

The key differences between this approach and the standard approach to analytic function are (1) Basing all the arguments on the closure of the complex plane $\overline{\mathbb{C}}$ (the Riemann sphere) instead of the complex plane $\mathbb{C}$. (2) The different definition of singular points based on topology. (3) The treatment of mappings $\overline{\mathbb{C}} \to \overline{\mathbb{C}}$ as multivalued functions without restricting their domains. This requires a different interpretation of equations which usually involve single valued quantities.
My earlier work on algebraic functions when considered as multivalued functions $z \to w$ where z and w are in the Riemann Sphere $\mathbb{C} \cup \{\infty\} = \overline{\mathbb{C}}$ seemed to indicate that their topology determines them uniquely apart from a few parameters. The topology of something is all the properties of it that are not changed by any continuous stretching without breaking and has been described as rubber sheet geometry. More precisely, algebraic functions are determined by the behaviours of these functions at their singular points and their locations.

The main theme of this paper is to investigate how this extends to "analytic" functions $\overline{\mathbb{C}} \to \overline{\mathbb{C}}$ that can be multivalued. The precise definition of this is not yet clear but the symbol $\mathcal{A}$ will be used for the set of functions concerned. The topological definition of a singular point used in my earlier paper [2] has been replaced by an equivalent analytical one. With multivalued functions, equations involving them have to be treated differently. Many examples are studied then some general theory is developed. Treating them like equations for single-valued quantities results in equations satisfied by the common singular points for their solutions. The complete set of singular points almost determines an analytic function uniquely. The idea of the *special* solution of an equation arises that has the minimum number of singular points.

Apart from the above, the notion of a singular point is slightly changed from my earlier work: the very special function $f : z \to 1/z$ that motivated the introduction of the point ∞ described in [2] (the Riemann Sphere) so as to make it left-unique as well as right-unique, is now not considered to have a singular point because of this. The point $(0, \infty)$ is now called an inversion point of $f()$.

Another important theme, though not yet fully developed, is that functions in $\mathcal{A}$ form a very complex algebraic structure that extends the algebra of algebraic functions in [2] by adding to it extra closure operations i.e. the passage to the limit of a sequence of such functions, and the solutions of equations of any type. This allows differentiation and integration to be included. If something like induction could be done it might provide another way to prove propositions.

The [4] says 'Each analytic function is an "organically connected whole", which represents a "unique" function throughout its natural domain of existence.' and I think this is the approach that should be followed.

Functions in $\mathcal{A}$ are in general multivalued (i.e. are relations) and therefore the general theory of relations must play a major role. Specifically, the concept of $\succ$ on functions in $\mathcal{A}$ which could be read as "could start with" for example the function given by $\sin(z^6)$ could be defined starting from z^2 or from z^3 and then applying another function $\mathcal{A}$. This has its origin in relations generally, and for this reason some of its basic properties needed to be established before applying them to functions in $\mathcal{A}$. The layout of the paper is as follows:

Notations and terminology including concepts from relations on an arbitrary set
A description of the closure operations with examples
A motivating very simple example of the equation mentioned above
A look again at algebraic functions and characterising power functions
Examples of functions in $\mathcal{A}$ and characterising their singular and inversion points
Properties of singular points
Solutions of equations and *special* solutions

1.1 General notations and terminology for relations

Relations generalise the concept of a mapping or function to the multivalued case. For an arbitrary set S a relation on S is a subset of $S \times S$. It will be useful to collect a few results, terminology, and notations involving relations here. The usual logical symbols $/$(typed over what it applies to) or $\neg, \forall, \exists, \in, \wedge, \vee, \Rightarrow, \Leftrightarrow$ mean "not", "for all", "there exists", "in", "and", "or", "implies", and "if and only if" respectively. The Boolean values 0 representing "false" and 1 representing "true" will be used throughout and the following equivalences occur frequently

$$\begin{array}{rcl} A \vee B = 0 & \Leftrightarrow & A = 0 \wedge B = 0 \\ A \wedge B = 0 & \Leftrightarrow & A = 0 \vee B = 0 \\ A \vee B = 1 & \Leftrightarrow & A = 1 \vee B = 1 \\ A \wedge B = 1 & \Leftrightarrow & A = 1 \wedge B = 1 \end{array} \tag{1}$$

for any Boolean variables A and B. Note that equality of relations is the same as logical equivalence often written as $\Leftrightarrow$.

A relation R is left-total if $\forall b \in S\, \{\exists a \in S[aRb]\}$ and likewise right-total if $\forall a \in S\, \{\exists b \in S[aRb]\}$. R is left-unique if $\forall a_1, a_2, b \in S[(a_1Rb \wedge a_2Rb) \Rightarrow a_1 = a_2]$ and likewise right-unique if $\forall a, b_1, b_2 \in S[(aRb_1 \wedge aRb_2) \Rightarrow b_1 = b_2]$. These meaning of these four terms seem to me to be immediately clear. The relation R is right-unique respectively left-unique $\Leftrightarrow$ R^{-1} is left-unique respectively right-unique, and likewise R is right-total respectively left-total $\Leftrightarrow$ R^{-1} is left-total respectively right-total. These terms replace the older terms: "one-to-one", "single-valued", "serial", "surjective" and "onto".

Apart from the operations of Boolean algebra that apply to all sets, extensive use will be made of composition (denoted by juxtaposition) and inversion. The composition of R_1 with R_2, R_1R_2 is defined by

$$\forall a, b \in S[aR_1R_2b \Leftrightarrow \exists c \in S[aR_1c \wedge cR_2b]]. \tag{2}$$

Note this is the logical order and corresponds (in the case of functions) to do R_1 then do R_2 i.e. $R_2(R_1(x))$. Then it follows that composition is associative and can be used to define the nth compositional power of a relation R for non-negative integers by $R^0 = I$ (the identity relation I is defined by $\forall a, b \in S[aIb \Leftrightarrow a = b]$), $R^1 = R$, and $R^{n+1} = R^nR$ and extend it to negative integers to give R^{-n} which is defined to be $(R^{-1})^n = (R^n)^{-1}$ which is also easily shown where the inverse of R written as R^{-1} is defined by $\forall a, b \in S[aRb \Leftrightarrow bR^{-1}a]$. (Note the corresponding operation for functions is written with an o before the integer exponent to distinguish this from the usual exponents, but this is not needed in the context of relations on an arbitrary set S. In case of confusion I will put the o back in).

Next follows two results that relate left-uniqueness to composition.

Lemma 1.1. *If R_1 is not left-unique and R_2 is left-total then R_1R_2 is not left-unique.*

Proof. If R_1 is not left-unique then

$$\exists a, b, c \in S[a \neq b, aR_1c, bR_1c].$$

Also if R_2 is left-total then

$$\forall e \in S[\exists d \in S[eR_2d]],$$

so choose $e = c$ then there exists a, b, c, d such that aR_1c, bR_1c, cR_2d and $a \neq b$ so aR_1R_2d and bR_1R_2d so R_1R_2 is not left-unique. □

Lemma 1.2. *If R_1 is right-unique and R_1 is right-total and R_2 is not left-unique then R_1R_2 is not left-unique.*

Proof. R_2 is not left-unique i.e.

$$\exists a, b, d \in S[a \neq b, aR_2d, bR_2d]$$

and R_1 is right-total i.e.

$$\forall f \in S[\exists e \in S[eR_1f]],$$

so choose $f = a$ then

$$\exists e_1 \in S[e_1R_1a]$$

and choose $f = b$ then

$$\exists e_2 \in S[e_2 R_1 b]$$

so $e_1 R_1 R_2 d$ and $e_2 R_1 R_2 d$. But if R_1 is right-unique

$$\forall a, b, c \in S[(aR_1 b, aR_1 c) \Rightarrow b = c]$$

so if $e_1 = e_2$ then $a = b$ which is not true therefore $e_1 \neq e_2$. This shows that $R_1 R_2$ is not left-unique. □

By introducing the inverses of these relations, equivalent results can be obtained.

The empty relation where $R = \emptyset$ satisfies $\forall a, b \in S[\neg aRb]$ and similarly $\forall a, b \in S[aRb]$ defines the total relation 1 and the negation symbol will also be applied to relations giving their complement so that $R \cup \neg R = 1$ and $R \cap \neg R = \emptyset$. The following properties hold $(R_1 \cup R_2)R_3 = R_1 R_3 \cup R_2 R_3$. $(R^{-1})^{-1} = R$, $(R_1 R_2)^{-1} = R_2^{-1} R_1^{-1}$. There is a general relation on relations I denoted by $\succ$, that has a simple definition based on composition,

$$R_1 \succ R_2 \Leftrightarrow \exists R_3[R_1 = R_2 R_3] \tag{3}$$

and could be read as "is more or equally complex as" or "can start with", which will probably not be clear until analytic functions (that actually are relations) are discussed. Its inverse could be denoted by $\prec$ means "is simpler or equally complex as". Any relation R satisfies

$$R \succ I \tag{4}$$

even if R is the empty relation $\emptyset$, and $\emptyset \succ R$ for any relation R. It is clearly reflexive ($R \succ R$) and transitive i.e. $(R_1 \succ R_2) \wedge (R_2 \succ R_3) \Rightarrow (R_1 \succ R_3)$.

1.2 Functions on the Riemann Sphere

This is an attempt to extend the treatment from what I defined as algebraic functions on the Riemann Sphere to all such functions in some sense.

In [2] the point ∞ was added to the complex plane to get the Riemann Sphere so that functions always have a value. This works for algebraic functions where continuity and differentiability hold for a function $f()$ even if $f()$ and its derivative go to ∞ there for example $z \to z^{-p/q}$ for $p, q \in \mathbb{N}, q \neq 0$ at $z = 0$. However this does have some unusual consequences for example $z \to \exp(1/z^2)$ at $z = 0$ which is 0 and ∞ because $\exp(\infty)$ is 0 and ∞ (this follows from $e^z = e^x e^{iy}$ if $z = x + iy$ where if x and y approach ∞ with x/y is constant, the result is 0 if $x \to -\infty$ and ∞ if $x \to \infty$, ∞ and $-\infty$ are the same point approached from opposite directions). These are examples of essential singular points for non-algebraic functions where the number of terms in the series about the singular points is infinite (Laurent series for finite singular points and power series for singular points at ∞).

Next follows a result that seems so fundamental that it should be perhaps mentioned here. It is connected with the completion of the complex plane to the Riemann Sphere $\overline{\mathbb{C}}$ and although I am probably not able to express or prove it properly, I present it as a theorem. Consider a function in $\mathcal{A}$ $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ with a single singular point at z_0 and a circuit described in z that is close to z_0 in $\overline{\mathbb{C}}$. The image of this not a circuit in $f(z)$ that is described just once in the same direction. Suppose there are no other singular points in $f()$ then this can be continuously deformed past ∞ without changing the discrete topology to a small circuit in z at some other point z_1 with a corresponding image in $f(z)$ where again the result is not a circuit described just once in the same direction. This small circuit in z can be made as small as you like while not crossing any singular or inversion point with the same result, and this would imply a singular or inversion point at $z_1 \neq z_0$ of the corresponding type. This contradiction proves that

Theorem 1.3. *An function in $\mathcal{A}$ defined on the Riemann Sphere $\overline{\mathbb{C}}$ cannot have only one singular or inversion point.*

Roughly this includes is any function, single or multivalued, that can be expressed by a formula that does not involve splitting the complex variable z into parts e.g. real and imaginary or modulus-argument etc. or is the solution of any problem defined using calculus involving such functions. See the closure operations below. They are differentiable and therefore infinitely many times differentiable in the extended sense (including ∞) wherever they are defined. They have no boundaries. The main difficulty with my approach compared with the standard approach to complex analysis is how to deal with multivaluedness. They are generally multivalued which can cause confusion as the examples show. This also affects how equations involving these functions are handled. Such equations frequently characterise singular points which is a major theme. There are closure operations that generate new functions from old ones and they start from the constant function and being infinitely differentiable i.e. analytic, so this term is used. The phrase analytic relations could be used because they can be multivalued, but I will stick to using the term analytic functions because of its common use. The term "analytic" is used because these functions in $\mathcal{A}$ will be closely related to complex analytic functions as this term is usually used.

The function exp() plays a very special role. It uniquely satisfies $\exp(0) = 1$ and $\exp'(z) = \exp(z)$. It satisfies $\exp(z) =$ the positive real value of e^z whenever $z \in \mathbb{Z}$ and e is the base of natural logarithms, and $\exp(x)$ is equal to the positive real value of e^x for other real x and is $e^x(\cos(y) + i\sin(y))$ when $z = x + iy$ thus there is a distinction between e^z and $\exp(z)$ with only the former being multivalued for non-integer and finite values of z. However due to the common usage that these are the same, if there is not likely to be an ambiguity e^z will be used when more properly $\exp(z)$ should be used.

Together with its inverse ln(), exp() can be used to define the general exponent function by

$$a^b = \exp(b \ln(a)). \tag{5}$$

To show that this in general has the correct number of values (q where $b = p/q$ and $\gcd(p, q) = 1$ i.e. p/q is in the lowest terms possible, with $p \in \mathbb{Z}$, $q \in \mathbb{N}$, $q > 0$), let $n \in \mathbb{N}$ with $0 \le n \le q-1$. Upon dividing np by q let $np = sq + r$ where $r \in \mathbb{N}$ and $0 \le r \le q-1$, and $s \in \mathbb{Z}$. $n_1 p \bmod q = n_2 p \bmod q \Rightarrow (n_1 - n_2)p = tq$ where $t \in \mathbb{Z}$. From this it follows that $q|(n_1 - n_2)p$ and because $q \nmid p$ it follows that $q|(n_1 - n_2)$ so $n_1 = n_2$ because n_1 and n_2 are in the range $0 \le n_1, n_2 \le q-1$ and $|n_1 - n_2| \le q-1$. Therefore the mapping $k()$ defined by $k : n \to np \bmod q$ is left-unique and generates a permutation of the integers $Q = \{0, 1, \ldots q-1\}$.

Therefore the fractional part of $\{np/q\}$ for $n \in Q$ is $\{n/q\}$ for $n \in Q$ in a different order and the sets $\exp(2\pi inp/q)$ and $\exp(2\pi in/q)$ where $n \in Q$ are the same but appear in a different order. Therefore the set of values of $\exp(b\ln(a))$ for one particular value of $\ln(a)$ is $\exp(\frac{p}{q}(\ln(a) + 2\pi in))$ where $b = p/q$ and is $\exp(\frac{p}{q}\ln(a))\exp(2\pi inp/q) = \exp(\frac{p}{q}\ln(a))\exp(2\pi in/q)$ therefore the expression $\exp(b\ln(a))$ has all q values and no others and can be used to define a^b.

A peculiar consequence of dealing with multivalued expressions is an ambiguity that can arise when doing calculations that involve them. Consider the following paradox which is probably one of the simplest examples of its kind:

$$e^{i\pi} = -1 \Rightarrow 2\pi i = 2\ln(-1) = \ln((-1)^2) = \ln(1) = 0! \tag{6}$$

While forgetting that ln() is multivalued it is too easy to carry out calculations like this and arrive at absurd conclusions. If for each instance of ln() it is remembered that any multiple of $2\pi i$ can be added to a result to give another value of the function, the following results are obtained:

$$\begin{aligned} &2\ln(-1) = 2(\pi i + 2n_1\pi i) = 2\pi i(1 + n_1) \\ &\ln((-1)^2) = 2n_2\pi i \end{aligned} \tag{7}$$

which are the same where n_1, n_2 are arbitrary integers. The logic is faulty in (6) where a multivalued expression is treated as a single value. Consider the generalisation $\ln(a^b) = b\ln(a)$. If $b\ln(a)$ represents one particular value of this multivalued expression, the complete set of values can be written as using $2\pi in_1 + b(\ln(a) + 2\pi in_2)$ i.e. $b\ln(a) + 2\pi i(n_1 + bn_2)$ for all $n_1, n_2 \in \mathbb{Z}$ where the expression $2\pi in_1$ can be added because it is the ln of something and the expression $2\pi in_2$ can be added because $\ln(a)$ is multivalued in the same way.

Therefore the complete set of values of $b\ln(a)$ can be written as

$$b\ln(a)+2\pi i(n_1+bn_2)=b\ln(a)+2\pi i\left(\frac{qn_1+pn_2}{q}\right). \tag{8}$$

where $b=p/q$ and p and q can be chosen to be coprime, so the numerator in the parenthises can be 1 by appropriate choice of n_1 and n_2, therefore it can be any integer by multiplying both n_1 and n_2 by that integer and thus $\exp(b\ln(a))$ has the q distinct values it should have. If $2\pi i n_1$ had not been added, so that $n_1=0$, pn_2 can be divided by q to obtain integers r and s such that $pn_2=rq+s$ where $0\le s<q$ then (8) would have been

$$b\ln(a)+2\pi i(r+s/q) \tag{9}$$

and s would have still attained all its values because of the following theorem in [3]:

Theorem 1.4. *If $a,b,c\in\mathbb{Z}$ then the equation $ax+by=c$ has a solution for $x,y\in\mathbb{Z}$ if and only if the greatest common divisor of a and b divides c.*

Therefore there is no problem with the multivalued nature of a^b if b is rational except if the exponent is regarded as a repeated multiplication when b is an integer. For example consider the expression $\left(\frac{1}{2}+\left(\frac{9}{4}\right)^{\frac{1}{2}}\right)^b$. The inner expression has values -1 and 2, so taking all possible values gives $(-1)^k 2^{(b-k)}$ for $0\le k<b$ whereas just the values $(-1)^b, 2^b$ should occur. A related example is what are the values of $1^{1/2}+1^{1/2}=(\pm1)+(\pm1)$? If these two instances have to be the same the result is ±2 otherwise 0 can be included. The general principle it seems to me is to take note of when two or more instances of the same multivalued expression occur in a formula have a common origin then they have to have the same value, otherwise they are independent.

2 Defining the algebra of functions

The set of algebraic functions as defined in [2] includes the constant functions $z\to c$ for any $c\in\overline{\mathbb{C}}$ and is closed under the following unary and binary operations on functions: union, composition, inversion, addition, subtraction, multiplication, division, differentiation with the exception that the inverse of the constant functions do not exist. The identity function $z\to z$ obtained by integrating the constant function equal to 1. The subtraction operation is merely the addition of a negative and so is not strictly required. The inclusion of division is needed to ensure that the special function $z\to 1/z$ is included.

The arithmetic operations just refer to the operations $f(z)=g_1(z)*g_2(z)$ for defining $f()$ in terms of $g_1()$ and $g_2()$ where $g_1()$ and $g_2()$ are functions in $\mathcal{A}$, then so will $f()$ where $*$ is $+$, $-$, $\times$ or $\div$.

The absence of integration as a closure operation for algebraic functions suggests the extension of these ideas to include it as an operation that gives closure. This requires the familiar functions ln() as the integral of $1/z$ and its inverse exp() to be included.

However including instead the limit of a sequence of functions can replace including derivatives and integrals. Then differentiation does not need to be included as a closure operation because a derivative is the limit

$$f'(z) = \lim_{h \to 0} \frac{f(z+h) - f(z)}{h} \tag{10}$$

of a difference that is already included. Also an integral is just the limit of a sum

$$\int_a^z f(t)dt = \lim_{n \to \infty} \left\{ \frac{1}{n} \sum_{i=0}^{n-1} f[a + i\left(\frac{z-a}{n}\right)] \right\} \tag{11}$$

which is already included.

It is also desirable to include all functions that are definable as solutions to equations of any type such as differential equations, integro-differential equations, difference equations. There are some examples later in the paper.

Therefore the closure operations involved the set of functions in $\mathcal{A}$ are as follows:

- union ($\cup$) i.e. from a set of functions, their union is found i.e. the graph of the union is just the set-theoretic union of the graphs of the separate function.
- composition and inversion (o or juxtaposition,$^{o-1}$)
- the four arithmetic operations $(+, -, \times, \div)$
- taking the limiting value of a sequence of functions
- solution of equations needs to be as general as possible and must include obtaining the special solution for $f()$ from $g_1()$ in (29) and obtaining the special solution for $f()$ from $g_2()$ in (30).

I was hoping that something like a universal base for computability might arise if this algebra could be made complete and correct.

Closure is a very attractive concept because it is possible to prove a proposition for every element of the algebra by a kind of induction by proving it for an initial set of special elements and proving that for every closure operation, it holds for the result of the closure operation if it holds for the elements to which the closure operation is applied. In this case if P is any proposition true for all elements of the algebra if and only if P is true for the special elements (the identity and constant functions).

$$\begin{array}{l} P[f_1()] \wedge P[f_2()] \Rightarrow P[f_1() \cup f_2()] \\ P[f()] \Rightarrow P[f()^{o-1}] \\ P[f_1()] \wedge P[f_2()] \Rightarrow P[f_1(f_2())] \\ P[f_1()] \wedge P[f_2()] \Rightarrow P[f_1() * f_2()] \text{ where } * = +, -, \times, \div \\ \forall i \in \mathbb{N}\{P[f_i()]\} \Rightarrow P[\lim_{i\to\infty} f_i()] \\ P[f()] \Rightarrow P[S[f()]] \end{array} \tag{12}$$

The first three involve only sets and relations. The limit operation allows all operations of calculus to operate within this algebra. Finally the solutions of equations includes simultaneous equations of any type and of course all these should be interpreted recursively so eg any equation involving given functions that are defined as above has solutions that are to be included etc.

Note that the limit of a set of continuous functions can be discontinuous in the real domain (Fourier series provide many examples), and this extends to evaluating a function in $\mathcal{A}$ on a path in $\overline{\mathbb{C}}$ that goes through a singular point that arises as a result of the limit taken.

Functions in $\mathcal{A}$ with a line of discontinuity can be extended i.e. analytic continuation ([1] Chapter 12) can be applied to extend the function on both sides of the boundary resulting in a multivalued function where this line of discontinuity is removed.

The obvious step is to define composition and inversion as for binary relations in general. This gives rise to compositional powers of functions eg $f(f(z)) = f^{o2}(z)$ defined as for relations in general. The symbol o is used to indicate the compositional power that follows it because o is sometimes used to indicate composition, and this distinguishes the inverse of a function from its reciprocal.

When working with multivalued functions, the equivalent of the function value is now a set of values and equality between relations is of course the equality between the two sets of values. This has consequences when manipulating equations with multi-valued functions in $\mathcal{A}$.

Perhaps this simplest closure operation is that of union. A union is simply the union of the two sets of pairs (z, w) defining each of the functions in the union. The concept of a union was not mentioned much in my previous paper. The simplest example of a union is when $f(z) = (z^2)^{1/2}$ which is the union of z and $-z$ which consists of pairs (z, z) and $(z, -z)$ for all z in $\overline{\mathbb{C}}$.

Suppose a single component function in $\mathcal{A}$ maps p values each to the same q values $\in \overline{\mathbb{C}}$. Does every single component function in $\mathcal{A}$ have to be like this, with p or q allowed to be ∞? In the two set of values, each member of a set is equivalent to any other member.

An function $f()$ in $\mathcal{A}$ has a single component if and only if for every pair of points P_1 and P_2 in $\overline{\mathbb{C}} \times \overline{\mathbb{C}}$ in the graph of $f()$ there is a continuous and analytic curve starting at P_1 and ending at P_2 at each point being in the graph of $f()$ and not including any singular point of $f()$ i.e. every such point is connected not via singular points to every other such point within the graph of $f()$.

Related to "union" is the concept of a component. A component will be a single analytic surface i.e. a function in $\mathcal{A}$ that itself could be multivalued. The number of components an function in $\mathcal{A}$ has will be an important property of it. Generally, only solutions of equations which consist of a single component are likely to be of interest. If a set of single components each satisfy an equation of the type considered here, then so does their union. Unless otherwise stated an arbitrary function will refer to a single component. The operation of extracting all the components from a union will probably be needed.

An function in $\mathcal{A}$ can be a union of smoothly differentiable components that each consist of a single continuum of points $(z, f(z)) \in \overline{\mathbb{C}} \times \overline{\mathbb{C}}$ provided there is an extension of the notion of differentiation from $\mathbb{C}$ to $\overline{\mathbb{C}}$. Finite and countable unions will surely be needed.

Singular points specified by $(z, f(z))$ are points in the analytic surface where a small circuit round z is *not* mapped into a small circuit round $f(z)$ in the Riemann Sphere. Another way to say this is that a singular point $(z, f(z))$ is any point about which for all neighbourhoods N of $(z, f(z))$ in $\overline{\mathbb{C}} \times \overline{\mathbb{C}}$ however small, the graph of $f()$ intersected with N is not topologically equivalent to an open disk. In such a case one value of z will correspond to more than one value of $f(z)$ or vice versa in N, see Section 6. Importantly, singular points are not to be confused with points where $f(z)$ is ∞ though these may often coincide. Definition: A singular point at (z, w) is finite iff $z \neq \infty$.

3 A simple example

Consider about the simplest example of an equation

$$f(z) = f(-z) \tag{13}$$

for a right-unique function $f()$. This is satisfied by $f_1(z) = z^2$ and by $f_2(z) = z^4$ and in fact any function of z^2. This suggests that the solution $f(z) = z^2$ has special significance and will be called a *special* solution.

Equation 13 can be written as $z_1 = -z_2 \Rightarrow f(z_1) = f(z_2)$. Suppose the condition 13 is required to be an inequality unless equality is explicitly required, then in the above case

$$f(z_1) = f(z_2) \Leftrightarrow z_1 = \pm z_2. \tag{14}$$

This strengthened condition eliminates z^4 from being a solution because then $f(z_1) = f(z_2) \Leftrightarrow z_1 = \pm z_2$ or $z_1 = \pm i z_2$ but the *special* solution of (13) does satisfy (14).

Singular points are defined to be points $P = (z_1, w_1 = f(z_1))$ where the function $f()$ is not behaving on any very small length scale surrounding P as a simple one to one correspondence. Thus a singular point is defined such that the function is not locally 1 to 1 there. Then minimising the number of singular points requires $f()$ to be locally 1 to 1 wherever possible. In this case it is not possible to have $f(z_1) = f(z_2) \Leftrightarrow z_1 = z_2$ (which would imply no singular points anywhere) because of (13), but (14) is next best because the possible arguments (and singular points) have to be treated in pairs $(z, -z)$.

The condition for the absence of a singular point at P is that for some neighbourhood N of P

$$z_1 = z_2 \Leftrightarrow f(z_1) = f(z_2), \tag{15}$$

holds for all $((z_1, f(z_1))$ and $z_2, f(z_2) \in N$. Using (14) this gives $z_1 = z_2 \Rightarrow f(z_1) = f(z_2) \Rightarrow f(z_1) = f(-z_2) \Rightarrow z_1 = -z_2$ which implies $z_1 = z_2 = 0$ or ∞. Therefore the singular points in solutions of (14) are only at $z = 0$ and ∞ which is where z^2 has singular points which shows that the number of singular points for solutions of (13) has been minimised by using instead the condition (14). Note Theorem (1.3) shows that no analytic function can have just one singular point.

Now suppose $f()$ satisfies 14, then introducing the function $k()$ by $k(z) = f(z^{1/2})$ then (14) is equivalent to $z_1 = z_2 \Leftrightarrow z_1^{1/2} = \pm z_2^{1/2} \Leftrightarrow f(z_1^{1/2}) = f(\pm z_2^{1/2}) \Leftrightarrow f(z_1^{1/2}) = f(z_2^{1/2}) \Leftrightarrow k(z_1) = k(z_2)$. Therefore the condition for the absence of a singular point for the function $k()$ holds everywhere. Therefore $k(z) = \frac{a+bz}{c+dz}$ by Lemma 4.2 implying $f(z^{1/2}) = \frac{a+bz}{c+dz}$ so $f(z) = \frac{a+bz^2}{c+dz^2}$.

4 basic theory

Theorem 4.1. *Every function $f() \in \mathcal{A}$ reaches every value $f(z) \in \overline{\mathbb{C}}$ i.e. is right-total for some $z \in \overline{\mathbb{C}}$ unless $f()$ is a constant function.*

Proof. This follows from the corresponding property of algebraic functions ($P(z, w) = 0$ always has a solution for z given w for any bivariate polynomial P) and the fact that functions $f() \in \mathcal{A}$ are continuous and are limits of sequences of algebraic functions which are all continuous. □

Closure under inversion requires every function in $\mathcal{A}$ to be left-total too. An interesting case occurs if the point that is the solution of such an equation approaches, under the limit, a singular point of the limit function. For example if the limit function is $f(z) = \exp(1/z)$ and the solutions approach $z = 0$ as would happen if $w = 0$. This works because $f(0)$ is 0 and ∞ i.e. both these values are attained by $f()$.

Lemma 4.2. *An function in $\mathcal{A}$ with no singular points and no inversion points is a linear function.*

Proof. The absence of a singular point at $z = \infty$ for a function $f()$ implies a neighbourhood of ∞ (a large circle in the complex plane but a small circle in the Riemann Sphere) in z maps in a left-unique manner locally to a neighbourhood in w say centred on $w_0 = f(\infty)$. If $w_0 \neq \infty$ then $1/z \approx a(w - w_0)$ for very large $|z|$ therefore $dw/dz = -1/az^2 \to 0$ as $z \to \infty$. Similarly if $w_0 = \infty$ a small neighbourhood in $1/z$ about 0 maps to a small neighbourhood round $1/w$ at 0 so $1/z \approx b/w$ therefore $dw/dz = b$ at (∞, ∞) and if there are no singular points and no inversion points anywhere in $w(z)$ then dw/dz is also everywhere finite and analytic there, so by Liouville's theorem (see for example [1]) dw/dz is constant so $w = a + bz$ where a and b are constants. □

Theorem 4.3. *A function in $\mathcal{A}$ with no singular points is a bilinear function given by $f(z) = \frac{a+bz}{c+dz}$.*

Proof. Let $f(z) = w$ be an function in $\mathcal{A}$ with no singular points. Then apply a bilinear function $b()$ to w such that $b(f(0)) = 0, b(f(1)) = 1, b(f(\infty)) = \infty$. This can be done uniquely (see [1] section 33). Then by Lemma 6.6 $b(f())$ has no singular points and maps, $0 \to 0$, $1 \to 1$, and $\infty \to \infty$. Also $b(f())$ can have no inversion point because if some finite point $z_0 \to \infty$ then $b(f())$ would be not left-unique there and would have a singular point there by Lemma 6.5 contradicting the assumption. Therefore $b(f())$ satisfies the condition of Lemma 4.2 and must be a linear function i.e. $b(f(z)) = \alpha + \beta z$ and $f(z) = b^{o-1}(\alpha + \beta z)$ which is also a bilinear function. □

The changed definitions of singular points and the new definition of an inversion point require some well-known theorems to be rephrased.

This is because "analytic" in the textbooks should be replaced by "right-unique analytic and finite" in the terminology of this paper. This would make the statements of theorems like the Cauchy integral formula slightly more cumbersome. Also "entire" means "right-unique, analytic, and without any singular points except possibly at $z = \infty$". If a function is bounded i.e. $|f(z)| < k$ for some $k > 0$ for all $z \in \mathbb{C}$ then by continuity, it cannot be ∞ at any point in $\overline{\mathbb{C}}$ including at ∞ itself. Therefore Liouville's theorem can be expressed as

Theorem 4.4. *If $f()$ is right-unique in $\mathcal{A}$, finite at every point $z \in \overline{\mathbb{C}}$ and without a singular point at any point $z \in \mathbb{C}$ then $f()$ is constant $\in \mathbb{C}$.*

Now suppose that $f()$ is right-unique, analytic and none of its values are equal to $w \in \mathbb{C}$ at any point $z \in \overline{\mathbb{C}}$ and $f()$ has no singular points with $z \in \mathbb{C}$. Then $\frac{1}{f(z)-w}$ is everywhere finite because $f(z)$ cannot approach w arbitrarily closely (for otherwise at the limit point it would equal w and $\overline{\mathbb{C}}$ is compact so includes all its limit points) and analytic and has no singular points for $z \in \mathbb{C}$, then by Theorem 4.4, $\frac{1}{f(z)-w} = c \in \mathbb{C}$, therefore $f(z)$ is constant $\in \overline{\mathbb{C}}$. This proves that

Theorem 4.5. *Every right-unique function $f() \in \mathcal{A}$ without any singular points where $z \in \mathbb{C}$, reaches every value $f(z) \in \overline{\mathbb{C}}$ for some $z \in \overline{\mathbb{C}}$ unless $f()$ is a constant $\in \overline{\mathbb{C}}$.*

5 Characterising singular points of algebraic functions

There seems to be much repetition from the old paper here! [

The topology of an algebraic function clearly must involve the behaviour at points that are not regular points where the behaviour is non-trivial. A simple way to describe this is to imagine a small circle described around the point (z_0, w_0) within the surface. Imagine it so small that no other points with irregular behaviour are included. If this can be done it will have projections down to both the z and w planes and if the circuit is complete ending where it started, the projections will be circuits around z_0 and w_0 respectively described p and q times, or for non-algebraic functions, either p or q may be infinite if the corresponding circuit never joins up again. Such points (z_0, w_0) with either p or q not equal to 1 are singular points and if $p = q = 1$ the point is a regular or non-singular point. Another kind of thing that can happen is when (z_0, w_0) is at the intersection of two or more surfaces, which again implies (z_0, w_0) is a singular point. In general a singular point is where in a small region surrounding it, the function surface(s) cannot be stretched so that it becomes flat.

Using the methods I developed earlier [2] to locate singular points for algebraic functions, suppose $w = z^{p/q}$ where $p, q \in \mathbb{N}$ then $w^q = z^p$ and $P = w^q - z^p = 0$ and $\partial P/\partial z = -pz^{p-1} = 0 \Rightarrow z^{p-1} = 0$ which is false if $p = 1$. If $p > 1$ then $z = 0$ and $w = 0$. Also $\partial P/\partial w = qw^{q-1} = 0 \Rightarrow w^{q-1} = 0$ which is false if $q = 1$. Because $\frac{\partial P}{\partial z} + \frac{\partial P}{\partial w}\frac{dw}{dz} = 0$ these conditions are equivalent to $\frac{dw}{dz} = 0$ or ∞. If $q > 1$ then $w = 0$ and $z = 0$. Therefore all finite singular points are at $(0, 0)$ provided $p > 1$ or $q > 1$ with the transformation $w^* = 1/w, z^* = 1/z$ giving the other one at $z^* = 0$, $w^* = 0$ i.e. (∞, ∞). Now suppose $p > 0$ and $q < 0$ then the same argument gives that all singular points are at $(0, \infty)$ or $(\infty, 0)$. In many examples of algebraic functions I have studied, it is easy to miss a singularity with either z or w being ∞ in addition to the finite singular points. It is later proved that no function in $\mathcal{A}$ can have just one singular point.]

Consider $f(z) = z^{1/p}$ where p is an integer. Rather than describing this behaviour simply by saying that it is expressed by a "winding number", near the branch point at $z = 0$, the idea is to relate $f(z)$ to f evaluated at the "next" branch of the function obtained by tracking $f(z)$ continuously once round a small circle surrounding $z = 0$ described in the anticlockwise direction until the same point z is reached. This circuit in z will have to be described p times to get back to the same value of $f(z)$. This is because if $f(z) = z^{1/p} = r^{1/p}e^{i\theta/p}$ then $f(z)^p = z = re^{i\theta}$ with $0 \leq \theta \leq 2\pi p$. Let $g_2(z) = e^{2\pi i/p}z$ where p is a positive integer. Then $g_2(f(z)) = e^{2\pi i/p}r^{1/p}e^{i\theta/p} = r^{1/p}(e^{2\pi i}e^{i\theta})^{1/p} = r^{1/p}e^{i\theta/p} = f(z)$ i.e.

$$f(z) = e^{2\pi i/p}f(z). \tag{16}$$

In fact this equation, being an equation for a multivalued function, represents the equality of the two sets of values each being p in number, and the equation generates a permutation of those p values. Equality of the sets of values will be implied whenever an equality occurs between two multivalued expressions. This is a simple example of equations which now have to be treated differently because the expressions are multivalued. This relationship is a better way of describing this situation than in [2] because it just involves the right-unique function $g_2()$ and no mention of topological concepts that are not so easy to make precise. Thus equations involving multivalued functions clearly cannot be treated as though the functions are right-unique and equations can be written down that would only have trivial solutions if they were for right-unique quantities. For example from (16) one cannot simply deduce that $f(z) = 0$ by subtracting $f(z)$ from both sides and dividing by $e^{2\pi i/p} - 1$. Obviously it is the first of these that goes wrong. The reason is that there are then two instances of $f(z)$ on the left hand side and it is not clear that these are the same one therefore $f(z)-f(z)$ has to be the set of every possible difference between the values of $f(z)$. Therefore likewise any binary operation with the second operand being multivalued should be avoided because the results are not likely to be useful. However, well chosen functions could be applied to both sides of a multivalued equation and be more useful as the following examples show.

The function $f(z) = z^{1/p}$ is clearly not the only solution of (16) (for example $f(z) = az^{1/p}$ or $f(z) = z^{q/p}$). Raising (16) to the power p gives the tautology $f^p = f^p$ so there is nothing that can be said about f^p so every solution of (16) is the pth root of some function in $\mathcal{A}$ regardless of its other singularities. This could be written as $f(z) = h(z)^{1/p}$ for an arbitrary function $h()$ is the general solution of (16). Any such function has a p-fold branch point at all points where $f = h = 0$, and satisfies (16) because $(e^{2\pi i/p})^p = 1$. In fact the general solution can be written as the following union $\{f(z).e^{2\pi ij/p}$ for $0 \leq j < p\}$ which is $(f(z)^p)^{1/p}$. This obviously satisfies (16) and has p components in general because each component is mapped to the next one by multiplying by

$e^{2\pi i/p}$. It could have fewer if some of them join to others. It is this example that motivated the introduction of the concepts of a union and the components of an function in $\mathcal{A}$. A simple example is $f(z) = z + 1$ which is not a solution of (16) because it is not the p-th root of an function in $\mathcal{A}$. It is not the pth root of $(z+1)^p$ which is the union $\{e^{2\pi ij/p}(z+1)$ for $0 \le j \le p-1\}$. The result of this can be written more succinctly as follows.

Lemma 5.1. *Suppose a function $f() \in \mathcal{A}$ satisfies (16) for some $p \in \mathbb{N}$ then this is equivalent to*

$$f(z) = h(z)^{1/p} \tag{17}$$

for some function $h() \in \mathcal{A}$.

In this context a concept arises, made precise later, which could be called the *special* solution of an equation. The *special* solutions of (16) are solutions $f_s(z)$ such that any other solution $f(z)$ of (16) can be written as $f(z) = f_s(h(z))$, so $f_s(w) = w^{1/p}$. This term will always be italicised to indicate this meaning. As will be shown later, the singular point(s) associated with an equation like (16) is/are given by its solutions where this equation is treated as an equation for a single valued quantity i.e. in this case where $f(z) = 0$ or ∞. Later it turns out that the *special* solutions can be parameterised by three independent parameters or by a, b, c, d and are $f_s(z) = (\frac{a+bz}{c+dz})^{1/p}$. The absence of extra singular points is required in $f_s()$ because if there was such a singular point then this would result in a corresponding singular point in $f_s(h(z))$ for any $h()$ without a singular point at the corresponding location i.e. for almost every function $h()$. A similar result is the following

Lemma 5.2. *If $q \in \mathbb{N}$ where $q > 1$ then*

$$f(z) = f(e^{2\pi i/q} z) \tag{18}$$

for all $z \in \overline{\mathbb{C}}$ for some function $f() \in \mathcal{A}$ if and only if

$$f(z) = h(z^q) \tag{19}$$

for all $z \in \overline{\mathbb{C}}$ where $h()$ is some other function in $\mathcal{A}$.

Note: if the first step in computing $h(z)$ is to apply $z \to z^{1/q}$, all q values must be included, giving a result which is a union of q components.

Proof. Equation (18) implies all q values $e^{2\pi ij/q}z$ for $0 \le j \le q-1$ have the same value of $f()$ and z^q is the same for all these. Also the distinct sets $\{z, e^{2\pi i/q}z, e^{4\pi i/q}z, \ldots e^{(q-1)\pi i/q}z\}$ for all $z \in \overline{\mathbb{C}}$ have the union which is $\overline{\mathbb{C}}$ and are disjoint. Thus any solution of (18) on the Riemann Sphere $\overline{\mathbb{C}}$ is of the form (19) and any function of this form satisfies (18) because $f(e^{2\pi i/q}z) = h((e^{2\pi i/q}z)^q) = h((e^{2\pi i/q})^q z^q) = h(z^q) = f(z)$. □

Again an argument motivating the concept of the *special* solution follows which can be done easily by inverting (18) and its solution and defining the inverse of the special solution of (16) to give the special solution of (18). Inverting (18) gives (16) after renaming $f^{o-1}()$ as $f()$ where $p = -q$. Therefore its special solution includes $f_s(z) = z^{-q}$ in the 3 parameter family, and therefore also $f_s(z) = z^q$. Any solution of (18) is the inverse of (17) which is $h(z^q)$ after renaming $h^{o-1}()$ as $h()$.

5.1 Examples

5.1.1 Example 1

Suppose $f(z) = (z - z_0)^q$ where q is a positive integer. Here the only finite singular point is at $(z_0, 0)$. Introducing the variable s by $s = z - z_0$, and $f^*()$ by $f^*(s) = f(z) = s^q$ then $f^*()$ satisfies (18). Therefore expressing this in terms of f using the chain of equalities

$$f(z) = f^*(s) = f^*(e^{2\pi i/q}s) = f^*(e^{2\pi i/q}(z - z_0)) = f(e^{2\pi i/q}(z - z_0) + z_0) \quad (20)$$

i.e. $f()$ satisfies

$$f(z) = f(g_1(z)) \text{ where } g_1(z) = e^{2\pi i/q}(z - z_0) + z_0. \quad (21)$$

This relationship just involves the right-unique function $g_1()$.

5.1.2 Example 2

Suppose a multivalued function satisfies

$$f(z) = e^{2\pi i/p} f(e^{2\pi i/q} z) \quad (22)$$

which is a combination of (16) and (18) where $p, q \in \mathbb{N}$. It can also be reasoned as follows: (22) is equivalent to $f^*(z) = f^*(e^{2\pi i/q}z)$ where now $f^*(z) = (f(z))^p$ or equivalently (by Lemma 5.2) $f^*(z) = h(z^q)$ i.e.

$$f(z) = (h(z^q))^{1/p} \quad (23)$$

for some arbitrary function $h() \in \mathcal{A}$.

Suppose both (16) and (18) are satisfied. This is equivalent to both (17) and (19) holding. Then in (17) $h(z)$ must also be of the form $h(z) = h_1(z^q)$ for some $h_1() \in \mathcal{A}$ (if it was not then (19) could not hold). Then putting these together gives (23) which is equivalent to (22) as above, and to both (17) and (19). These equivalences can be written as

Theorem 5.3.

$$\begin{gathered} f(z) = e^{2\pi i/p} f(z) \wedge f(z) = f(e^{2\pi i/q} z) \Leftrightarrow \\ f(z) = h(z)^{1/p} \wedge f(z) = h(z^q) \Leftrightarrow \\ f(z) = (h(z^q))^{1/p} \Leftrightarrow \\ f(z) = e^{2\pi i/p} f(e^{2\pi i/q} z) \end{gathered} \tag{24}$$

This suggests that in general there are two auxiliary functions for an equation in $\mathcal{A}$ like (22) to be denoted by $f_{s1}()$ and $f_{s2}()$ such that the general solution to an equation such as (22) is $f_{s1}(h(f_{s2}(z)))$ where $h()$ is arbitrary in $\mathcal{A}$ and the *special* solutions of (22) have this form with $h()$ being a bilinear function giving the form

$$f(z) = \left(\frac{a + bz^q}{c + dz^q} \right)^{1/p}. \tag{25}$$

where in this case $f_{s1}(z) = z^{1/p}$ and $f_{s2}(z) = z^q$.

******** section checked to here 2025-04-19 ************

If there are other singularities, equations like (18) and (19) will not necessarily be exact but only asymptotically correct as the corresponding singular point is approached. For example in (25) if $z = (-b/a)^{1/q} + \epsilon$ then $f(z)$ can be expanded as a power series in ϵ in which terms higher than the first contribute so that the asymptotic behaviour near $((-b/a)^{1/p}, 0)$ is affected by the singular point at $(0, 0)$.

5.1.3 Example 3

The ideas in Equations (21) and (16) can be combined by considering the solutions of

$$f(z) = e^{2\pi i/p} f(e^{2\pi i/q}(z - z_0) + z_0). \tag{26}$$

Introducing the new variable $s = z - z_0$ and the new function $f^*(s) = f(s + z_0)$ then (26) becomes

$$f^*(s) = e^{2\pi i/p} f^*(e^{2\pi i/q} s) \tag{27}$$

whose general solution is $f^*(s) = (h(s^q))^{1/p}$ i.e. therefore the general solution of (26) is $f(z) = [h((z - z_0)^q)]^{1/p}$.]

As would be expected (and is justified later) the singular point(s) of $f()$ are given by

1. where the argument of the p-th root i.e. $h((z - z_0)^q) = 0$ or ∞
2. where $(z - z_0)^q$ is a singular point of $h()$
3. where $z - z_0$ is a singular point of the q-th power function which is at 0 and at ∞ so $z = z_0, \infty$.

For the case where $h()$ is the identity function, the second singular point no longer exists and the first and third of these singular points coincide at $z = z_0, \infty$ and $f(z) = (z - z_0)^{q/p}$ and the winding number ratio is $p : q$ in the earlier description.

6 Definition and properties of singular points

In the examples above the precise definition of singular points and many properties they have, have been hinted at. In this section these are described in the general context in which the function $f() \in \mathcal{A}$ is not necessarily an algebraic function. In all these definitions, a neighbourhood of a point $(z, w) \in \overline{\mathbb{C}} \times \overline{\mathbb{C}}$ is an open set containing (z, w) in the cartesian product topology.

These results depend on general properties of mappings right-unique versus multi-valued, and left-unique versus many-to-one. These properties can be defined such that they are local to a particular point as follows.

Definition 6.1. *The function $f()$ is locally left-unique at $P = (z, f(z))$ if and only if there is a neighbourhood N of P such that for every pair $(z_1, f(z_1))$ and $(z_2, f(z_2))$ in N, $z_1 \neq z_2 \Rightarrow f(z_1) \neq f(z_2)$.*

and likewise

Definition 6.2. *The function $f()$ is locally right-unique at $P = (z, f(z))$ if and only if there is a neighbourhood N of P such that for every pair $(z_1, f(z_1))$ and $(z_2, f(z_2))$ in N, $f(z_1) \neq f(z_2) \Rightarrow z_1 \neq z_2$.*

Definition 6.3. *$f()$ has a singular point P at $(z, f(z))$ if and only if for all neighbourhoods N of P there exists $(z_1, f(z_1)) \in N$ and $(z_2, f(z_2)) \in N$ such that either $[z_1 \neq z_2$ and $f(z_1) = f(z_2)]$ or $[z_1 = z_2$ and $f(z_1) \neq f(z_2)]$.*

A statement equivalent to definition (6.3) is to require this condition only for all neighbourhoods intersected with a specified neighbourhood of P however small it is. This makes it clearer that the condition is a local property of the behaviour at P. This is the same as saying the condition that needs to be satisfied for the absence of a singular point of the function $f()$ at the point P, $(z, f(z))$ is that there exists a neighbourhood N of P such that

$$\forall (z_1, f(z_1)), (z_2, f(z_2)) \in N[z_1 = z_2 \Leftrightarrow f(z_1) = f(z_2)] \tag{28}$$

i.e. $f()$ is left-unique and right-unique within N. This condition is very complicated to work with. A simpler equivalent form can be derived from it roughly as follows. Because there is no limit to how small the neighbourhood N can be (except that it cannot consist of the point P alone having no size because this is not an open set), one can write $z_1, z_2 \approx z$ because these can be made arbitrarily close to each other. By making N sufficiently small about P, the condition in square brackets can be made to be true unless the number of solutions of $f(z_1)-f(z_2)=0$ within N is just one i.e. $z_1=z_2$ at P and is more than one in N other than at P itself. This is because a very small neighbourhood N of P is then a single disc and so no more than one value of z_2 can be within N for a fixed z_1 and satisfy $f(z_1)=f(z_2)$. Therefore the whole condition for the *presence* of a singular point at P is false unless the number of solutions of $f(z_1)=f(z_2)$ changes at P. This can happen in two ways first when P is an intersection point of two surfaces in the graph of $f()$ (whether in a single component or not), and secondly when P is a point where the first derivative $f'(z)$ is 0 or ∞ which prevents $f()$ being one-to-one in an infinitesimally small disc surrounding P. In the first case, to find the singular points $(z_1, f(z_1))$, solve $f(z_1)=f(z_2)$ and $z_1=z_2$ simultaneously to find the non-trivial solution (it holds everywhere trivially) by first eliminating z_2 and introducing $f_1()$ and $f_2()$ as two branches of $f()$ to get $f_1(z_1)=f_2(z_1)$. This works provided these equations are not consistent everywhere which would happen if $f()$ was left-unique and right-unique which would make $f_1(z)=f_2(z)$ hold everywhere. Curiously there may be a connection between these ideas because in the case $f(z)=z^2$, it gives $0=f(z_1)-f(z_2)=z_1^2-z_2^2=(z_1+z_2)(z_1-z_2)$ giving $z_1=z_2$ as the trivial case, and the other case is when $z_1+z_2=0$ also, so $z_1=z_2=0$, and (0,0) is the finite singular point (the other one is at (∞,∞)). Therefore

Lemma 6.4. *in definition 6.3 the location of the singular point(s) is determined by (i) $f'(z)=0$ or ∞ or (ii) where any two of the analytic surfaces for $f()$ (called locally $f_1()$ and $f_2()$) cross i.e. where $f_1(z)=f_2(z)$. These are the non-trivial solutions of $z_1=z_2$ and $f(z_1)=f(z_2)$ (they are trivially true everywhere).*

Lemma 6.5. *A function $f()$ has a singular point at $P=(z,f(z))$ if and only if $f()$ is either not locally left-unique there or $f()$ is not locally right-unique there.*

Proof. It is only necessary to choose the neighbourhood that is the intersection of the two neighbourhoods in definitions 6.1 and 6.2 and take the negation of the result. □

Lemma 6.6. *Composition with a function $h()$ that is in $\mathcal{A}$ and has no singular point at a particular location implies that the singular/non-singular status of $f()$ is the same as that of $h(f())$ and $f(h())$ each at the corresponding point of $f()$.*

This is obvious from the assumption that any $h()\in\mathcal{A}$ is a smooth function and is locally one-to-one away from singular points.

Definition 6.7. *$f()$ has an inversion point at $(z, f(z))$ if and only if $f(z) = \infty$.*

It is possible for a singular point to also be an inversion point e.g. $f(z) = z^{-2}$ at $z = 0$. An example of an inversion point that is not a singular point is $f(z) = z^{-1}$ at $z = 0$ because this function is everywhere right-unique and left-unique.

The definition used in my earlier paper on algebraic functions [2] includes inversion points with the singular points, and inversion points were not considered as a separate category. The reason for separating them out is for consistency in definition 6.3 that now works even if $f(z) = \infty$ where a neighbourhood of ∞ is as would be expected on the Riemann Sphere i.e. a region of the complex plane outside of a finite connected region defined by a single boundary.

A topological argument involving moving $f(z_0)$ to ∞ where z_0 is a singular or inversion point suggests that the direction of traversal of $f(z)$ round a circuit surrounding $(z, f(z))$ (P) is the same as that of the corresponding circuit z for any point P in the graph of $f()$ except when $f(z_0) = \infty$ when it is reversed as the result of this circuit crossing ∞.

The following results relate singular behaviour to the operations of inversion, composition, arithmetic operations, and union.

Lemma 6.8. *$(z, f(z))$ is a singular point of $f()$ if and only if $(f(z), z)$ is a singular point of $f^{o-1}()$.*

Proof. Lemma (6.4) makes this obvious. □

For a very similar reason

Lemma 6.9. *Combining a function in $f() \in \mathcal{A}$ with another function in $g() \in \mathcal{A}$ without a singular point using $*$ will not alter the singular/non-singular status of $f()$ at the corresponding point where $* = +, -, \times$ or $\div$.*

This result is obvious:

Lemma 6.10. *the only singular points of a union that are not included in one of the separate components is where at least two components intersect.*

These are known as intersection singular points.

Lemma 6.11. *If $f(), h() \in \mathcal{A}$ and $f()$ is right-unique with a singular point at $(z_1, f(z_1))$ then $h(f())$ is singular point at the corresponding point(s) $(z_1, h(f(z_1)))$.*

Proof. Consider a neighbourhood N of $(z_1, h(f(z_1)))$. Then because of Theorem 4.1, there is a corresponding neighbourhood N' of $(z_1, f(z_1))$ under mapping by $h^{o-1}()$. Because $f()$ has a singular point at $(z_1, f(z_1))$, there exists $(z_2, f(z_2))$ and $(z_3, f(z_3)) \in N'$ such that $z_2 \neq z_3$ and $f(z_2) = f(z_3)$. This is because $f()$ is right-unique so the second option in definition 6.3 is not possible. Therefore there exists $(z_2, h(f(z_2)))$ and $(z_3, h(f(z_3))) \in N$ where $z_2 \neq z_3$ and $h(f(z_2)) = h(f(z_3))$. This works for any neighbourhood N implying $h(f())$ has a singular point at $(z_1, h(f(z_1)))$. □

This has a corresponding lemma obtained by inversion. Expressing everything in terms of the inverse functions gives

Lemma 6.12. *If $f^{o-1}(), h^{o-1}() \in \mathcal{A}$ and $f^{o-1}()$ is left-unique with a singular point at $(f(z_1), z_1)$ then $f^{o-1}(h^{o-1}())$ is singular point at the corresponding point(s) $(h(f(z_1)), z_1)$.*

Then renaming $f()$ and $h()$ to their inverses and making other changes of notation simplifies the presentation:

Lemma 6.13. *If $f(), h() \in \mathcal{A}$ and $f()$ is left-unique with a singular point at $(z, f(z))$ then $f(h())$ is singular at the corresponding point(s) $(h^{o-1}(z), f(z))$.*

The importance of Lemma 6.11 is that it is not possible to remove a singular point in a right-unique function in $\mathcal{A}$ e.g. $z \to z^2$ by applying another function to the result. For example applying $z \to z^{1/2}$ gives the union $z \to \pm z$ that has an intersection singular point where these components coincide at $(0, 0)$. The condition that $f()$ is right-unique is important, for example applying these functions in the other order gives $(z^{1/2})^2 = z$ without any singular points.

7 Solutions of the equations (29) and (30) defining singular points and types of multivalued functions

************** note to the reader of earlier versions: the roles of (29) and (30) have been reversed, and $g_1()$ and $g_2()$ have been exchanged and the presentation of the main theorems has been reversed to make the final presentation look right eg $g_1()$ is introduced before $g_2()$ etc. *******************

************* this section is still under construction ***********

My earlier thoughts about equations such as (29) and (30) is that they were just descriptions of the asymptotic behaviour of $f(z)$ close to the relevant singular point, but they can be equations that are satisfied exactly by certain multivalued analytic functions.

In the former approach, if there are other singular points whose behaviour is specified in their vicinity, then I expect (50) will only asymptotically hold close to the corresponding singular point. Such functions will not be solutions of (50) except asymptotically close to the appropriate singular point. It is interesting to consider how such functions might be constructed just from their singular behaviour at these points. A possibility is linear combination (LC) of the minimal i.e. special solutions for each separate singular point. This LC will have precisely the asymptotically defined behaviours at the singular points because all the other terms will not have a singular point at each of them. This I think can perhaps be generalised to nonlinear combinations, if the condition of minimality is dropped. i.e. find the general solution of a set of simultaneous asymptotically defined functions about a set of singular points as some arbitrary function in $\mathcal{A}$ of basic solutions to them singly?

For a multivalued single component function $f()$ in $\mathcal{A}$ it is possible to have a circuit in which the z value is returned to but w comes back to a different value. That gives rise to an equation of type (81). As the circuit is reduced in size, at some points the final value reached will suddenly change and will eventually will suddenly equal the original value. It suddenly changes where the curve crosses a singular point of which there can be many. The singular point is where the two values of $f()$ coincide. Having found all the singular points and their associated equations relating the function values, it should be possible to, by following any combination of the paths in any order allowing repetition, to get from say (z_1, w_1) to any other point (z_1, w) in the graph of $f()$. This would indicate that all the equations of type (81) have been found. It is possible (see for example (71)) that there is a pair (or perhaps more) of singular points that are associated with the same transformation (81) or its inverse.

Similarly there can be circuits that return the w to the same value but z returns to a different value. This gives rise to an equation of type (29) and is equivalent to doing the same thing for $f^{o-1}()$. There could be a finite or a countably or uncountably infinite number of singular points. See for example (67) with solution (69) that has uncountably many singular points on the unit circle. [Is this correct? For the case when the number of singular points is finite or countably infinite, this leads to the graph of $f()$ being described as a set of collections of points say $z_1, z_2 \dots z_p, w_1, w_2 \dots w_q$ such that every one of the z's is mapped to all of the w's in every collection. Away from singular points, all the z's are distinct and so are all the w's. Therefore the positive integers p, q are constants for the function $f()$, but either could be ∞. It may be useful to define the signature of an function in $f()$ to be say $\{(p_1, q_1), (p_2, q_2), \dots\}$ where each of the pairs corresponds to one component of the function.]

The fact the functions can be multivalued and can have many components does unfortunately introduce some complexity. Many components in a function might make it unclear how to count a set of functions. For example functions are different if their sets of components are different in any way e.g. it is possible to have a set of functions each of which has a single component. Also another set of functions could have all possible pairs of the same set of components with each pair in a different function, and another could have all possible non-empty subsets of the same set of components, each in a different function, and finally a single function that has all the components in the sets. For solutions of equations it will usually be the case that if a set of components satisfies the equations, any function that consists of a non-empty subset of the components will also satisfy them which of course includes the function that has all the components. This will be the case if the equation only involves the function values and not anything else such as the number of such values etc. and applies to equations (30) and (29) below. Much of this complexity can be mitigated by referring to the set of single-component solutions of an equation.

An equation $w = f(z)$ means that w is one of the values of $f(z)$ i.e. $w \in f(z)$ or w is the set of all values of $f(z)$ depending on context.

In Section 5 the singular points associated with power functions are all described by special cases of the following type of equations

$$f(z) = f(g_1(z)). \tag{29}$$

$$f(z) = g_2(f(z)) \tag{30}$$

These equations can be associated in general with functions that are not either left-unique or right-unique. For example if $f() \in \mathcal{A}$ is not right-unique so that it has many different values generally, then at one point z_1 by picking on one pair of values, the first and the second, another function $g_2()$ can be defined by mapping from the first to the second after letting z_1 vary smoothly over all of $\overline{\mathbb{C}}$ so that, in the absence of singular points of $f()$ where a pair of these values become equal, if $f()$ is continuous so is $g_2()$. Thus $g_2()$ is defined by (30). On circling a singular point the values may swap or change so in general $g_2()$ will be multivalued and may not be unique. If $g_2()$ is not unique (and trivially if it is) the members of $g_2()$ will form a group under composition because the composed function will still be a function mapping value(s) of $f()$ to others. Likewise $f()$ that is not left-unique will allow the function $g_1()$ to be constructed satisfying (29) and the set of such functions will form a group under composition.

Later on many examples arise in which these types of equation seem to not only characterise the singular points, but they also act as defining equations for classes of functions $f() \in \mathcal{A}$. This is also connected closely with their *special* solutions. Some basic results follow regarding equations (29) and (30) followed by examples that motivate some general theory of *special* solutions.

[Any single component multivalued function $f() \in \mathcal{A}$ can be a solution of (30) because once $f()$ is chosen, pairs of values $(w, g_2(w))$ are obtained by putting each possible value of $z \in \overline{\mathbb{C}}$ into (30) making sure that every pair of values $(z, f(z))$ of the multivalued $f(z)$ is included for each value of z. The graph of $g_2()$ (i.e. the set (z, w) such that $w = g_2(z)$) must be a subset of this i.e. one of its components. It is not necessary that every value of $f(z)$ is mapped to every other one by $g_2()$. If $f(z) = \{w_1, w_2, w_3\}$ it could be that $g_2()$ maps w_1 to w_2, and w_2 to w_3 and w_3 to w_1. For example (16) with $p = 3$.]

Theorem 7.1. *Any single component function $f() \in \mathcal{A}$ satisfies* (29) *and the associated single-component functions $g_1()$ form a group under composition.*

Inverting this gives $f^{o-1}(z) = g_1^{o-1}(f^{o-1}(z))$ where $g_1^{o-1}()$ is uniquely determined therefore by renaming $f^{o-1}()$ as $f()$ and renaming g_1^{o-1} as $g_2()$

Theorem 7.2. *Any single component function $f() \in \mathcal{A}$ satisfies* (30) *and the associated single-component functions $g_2()$ form a group under composition.*

Lemma 7.3. *If $f() \in \mathcal{A}$ satisfies* (30) *with $g_2()$, then $f()$ has singular points at every point (z, w) that is a solution of $w = g_2(w)$ where $w = f(z)$.*

Proof. If Lemma 6.4 is applied to solutions of (30) let $w_i = f(z)$, two branches of $f()$ are related by $w_1 = g_2(w_2)$ (amongst other possible relationships) and so their equality which is the meaning of Lemma 6.4 gives the equations $w = g_2(w)$ with $w = f(z)$ that define the locations of some of the singular points in solutions for $f()$ of (30) i.e. those common to all solutions $f()$. □

Next follows a few lemmas that are almost obvious which follow from properties of relations in general. They are needed to complete the main theorems of this section.

By Theorem (4.1), analytic functions are always total (left and right) except for the constant function, Lemmas (1.1) and (1.2) can be expressed in terms of functions as follows

These results are not related to (29) or (30) [

Lemma 7.4. *If $f_2(f_1())$ is left-unique then $f_1()$ is left-unique or $f_2()$ is constant.*

Lemma 7.5. *If $f_2(f_1())$ is left-unique then $f_2()$ is left-unique or $f_1()$ is not right-unique.*

Renaming $f_1()$ and $f_2()$ to their inverses and using the facts that inversion swaps left- and right- uniqueness and left- and right- totality, and the inverse of a constant function does not exist, gives two obvious results that can be combined as follows

Lemma 7.6. *If $f_1(f_2())$ is right-unique then (i) $f_1()$ is right-unique and (ii) either $f_2()$ is right-unique or $f_1()$ is not left-unique.*

]

Lemma 7.7. *If $f() \in \mathcal{A}$ satisfies (29) with $g_1()$, then $f()$ has singular points at every point (z, w) that is a solution of $z = g_1(z)$ where $w = f(z)$.*

Proof. Invert (29) to get $f^{o-1}(w) = g_1^{o-1}(f^{o-1}(w))$ and apply Lemma 7.3 and rename $f^{o-1}()$ as $f()$ and $g_1^{o-1}()$ as $g_2()$. □

Applying $h()$ to both sides of (29) shows that if $f()$ is a solution of (29) so is $h(f())$ for any $h() \in \mathcal{A}$. Starting with $z_2 = g_1(z_1)$, applying $f()$ to both sides and using (29) gives $f(z_2) = f(g_1(z_1)) = f(z_1)$ i.e. from equation (29) it follows that

$$z_2 = g_1(z_1) \Rightarrow f(z_1) = f(z_2). \tag{31}$$

From (31) it follows that, starting with $z_2 = g_1(z_1)$, gives

$$f(g_1(z_1)) = f(z_2) = f(z_1) \tag{32}$$

so (29) with $z = z_1$ holds which is independent of z_2 and (32) which will be true for one value of z_2 therefore (29) holds that does not depend on the value of z_2; this is the converse showing that (31) is equivalent to (29). With different symbols this is the same as

$$z_3 = g_1(z_2) \Rightarrow f(z_2) = f(z_3) \tag{33}$$

and substituting (31) into (33) gives $z_3 = g_1(g_1(z_1)) \Rightarrow f(z_1) = f(z_3)$. This can be repeated any number of times giving $z_2 = g_1^{on}(z_1) \Rightarrow f(z_1) = f(z_2)$ for any $n \in \mathbb{N}$. Because this is symmetric, exchanging z_1 and z_2 and combining the results, this can be written as $(z_1 = g_1^{on}(z_2)) \vee (z_2 = g_1^{on}(z_1)) \Rightarrow f(z_1) = f(z_2)$ and

$$\exists n \in \mathbb{N}[(z_1 = g_1^{on}(z_2)) \vee (z_2 = g_1^{on}(z_1))] \Rightarrow f(z_1) = f(z_2). \tag{34}$$

Suppose that the special solutions of (29) are defined to satisfy in addition the converse of this, it can be written as

$$f_{s1}(z_1) = f_{s1}(z_2) \Leftrightarrow \exists n \in \mathbb{N}[(z_1 = g_1^{on}(z_2)) \vee (z_2 = g_1^{on}(z_1))] \tag{35}$$

which can be abbreviated to $z_1 \sim z_2$ which is an equivalence relation.

This states that the distinct values of $f_{s1}()$ are in one to one correspondence with the equivalence classes of $\sim$. Any solution $f(z)$ of (29) is a function of the equivalence classes i.e. its value is the same for each member of the same equivalence class, therefore it can be written as a function of an arbitrary such function $f_{s1}()$ i.e. $f(z) = h(f_{s1}(z))$ where $h()$ is also right-unique by Lemma 7.6 if $f_{s1}()$ is. Also by Lemma 6.6, the singular points of $f()$ will include all points corresponding to the singular points of $f_{s1}()$ and of $h()$. Therefore the functions $f_{s1}()$ are solutions of (29) with the minimum number of singular points. Here clearly $f_{s1}()$ can be chosen to be single-valued i.e. right-unique. Intuitively, if $z_1 \neq g_1(z_1)$, so in the above notation, $g_1^{ok}(z_1)$ will all be different for all $k \in \mathbb{Z}$ so if z describes a small circle around z_1, the equivalence classes corresponding to each of these points have $z's$ that also do this and so the equivalence classes are a single valued function of z that is close to z_1. Therefore there is no singular point in $f_{s1}(z)$ where $z \neq g_1(z)$. A formal proof seems difficult. This is the meaning of the paragraph containing (82).

By its definition, $f_{s1}()$ is not actually a unique function. Any other function $f^*_{s1}()$ could be used in its place where $f^*_{s1}(z) = h(f_{s1}(z))$ where $h()$ is any left- and right-unique function which must be a bilinear function.

For any more complex cases such as when $f()$ has sets of values, these sets will then be in one to one correspondence with the equivalence classes of $\sim$ and this can be represented like this with a multivalued function $h()$. This shows that

Theorem 7.8. *A function $f() \in \mathcal{A}$ satisfies* (29) *with $g_1() \in \mathcal{A}\backslash I$ i.e. $g_1()$ in not the identity function I, if and only if $f(z) = h(f_{s1}(z))$ for some function $h() \in \mathcal{A}$ where $f_{s1}()$ is a right-unique solution of* (29) *with $g_1()$, such that in addition, $f_{s1}()$ has the minimum number of singular points i.e. singular points $(z, f(z))$ only where z satisfies $z = g_1(z)$. If $f()$ is right-unique so is $h()$.*

An example is $f(z) = f(2z)$. This leads to the equivalence relation

$$\exists n \in \mathbb{N}[z_1/z_2 = 2^n \text{ or } 2^{-n}] \Leftrightarrow \exists n \in \mathbb{Z}[\ln_2 z_1 - \ln_2 z_2 = n] \tag{36}$$

so $\Im(\ln_2(z_1) - \ln_2(z_2)) = 0$ and real part can be expressed in terms of the floor of these values i.e. these values rounded down to integers i.e. $\ln_2(z_1) - \ln_2(z_2) = \lfloor \Re \ln_2(z_1) \rfloor - \lfloor \Re \ln_2(z_2) \rfloor$ so the general solution of $f(z) = f(2z)$ is $h(f_{s1}(z))$ where $f_{s1}(z) = \ln_2(z) - \lfloor \Re \ln_2(z) \rfloor$ and $h()$ is arbitrary in $\mathcal{A}$. Here $\lfloor$ and $\rfloor$ bracket an argument of the “floor” function that rounds down to the closest integer.

If $f(z) = z^{1/2}$ then $f(z) = f(z^{2^{-n}})$ for all $n \in \mathbb{N}$ which converges to all the values on the unit circle which is common to all the equivalence classes. Therefore a smooth solution must be constant.

Definition 7.9. *With reference to Theorem 7.8, define $S[]$ as the mapping from $g_1()$ to $f_{s1}()$. It maps a group X of single-component functions $g_1()$ in $\mathcal{A}$ (typically containing just the identity and a function and its inverse and can be defined by giving its generators) to a set Y of members of $\mathcal{A}$. The set Y also forms a group under composition and is such that every member $f()$ of Y is a left-unique and right-unique function of every other member $f^*()$ of Y i.e. $f(z) = \frac{a+bf^*(z)}{c+df^*(z)}$ for some $a, b \in \overline{\mathbb{C}}$. Therefore the set Y is represented by any single member of it and this fact motivates the notation $S[g_1()] = f_{s1}()$ as though the result of $S[]$ has a single member. The square brackets are the* *notation for an operator i.e. a function with another function as an argument, and S can be thought of as an abbreviation for "simplest" or "special".*

The operator S is the mapping from a generator of the group X (if there is only one) e.g. $g()$ to any member of $S[g()]$. From (29) by substituting $z = g^{o-1}(w)$ or $w = g(z)$ it can be written as $f(g^{o-1}(w)) = f(w)$ which is an equation of the same form, and because (35) is the same when $g()$ is replaced by its inverse, it follows that

$$S[g()] = S[g^{o-1}()]. \tag{37}$$

Also by substituting $g^{ok}(z)$ for z in (29) gives $f_{s1}(g^{ok}(z)) = f_{s1}(g^{o(k+1)}(z))$ for any $k \in \mathbb{N}$. Then $S[g()] = l()$ implies $l(z) = l(g(z)) = l(g^{o2}(z)) \ldots = l(g^{on}(z))\ \forall n \in \mathbb{N}$. Also $l()$ only has singular points where $z = g(z)$ i.e. $l()$ has singular point at $(z, l(z))$ implies $z = g(z)$. From this it follows that $l()$ has singular point at $(z, l(z))$ implies $z = g^{on}(z)$ which is the same as $l()$ only has singular points at points $(z, l(z))$ such that $z = g^{on}(z)$ so $l()$ has both properties of a member of $S[g^{on}()]$. Therefore $S[g()] \subseteq S[g^{on}()]$. Then from the structure of the result of S i.e. the Y above and (37), it follows that $S[g()] = S[g^{on}()]$ for all $n \in \mathbb{Z}$ except $n = 0$ i.e. the following result is proved

Theorem 7.10.

$$\forall n \in \mathbb{Z}\backslash\{0\}\ \{S[g()] = S[g^{on}()]\} \tag{38}$$

From (35) it follows that

$$\begin{aligned} &z_1 \sim_k z_2 \Leftrightarrow \exists n \in \mathbb{N}[(z_1 = g_1^{onk}(z_2)) \vee (z_2 = g_1^{onk}(z_1))] \Rightarrow \\ &\exists n \in \mathbb{N}[(z_1 = g_1^{on}(z_2)) \vee (z_2 = g_1^{on}(z_1))] \Leftrightarrow z_1 \sim z_2 \end{aligned} \tag{39}$$

where $\sim_k$ has been introduced in a manner analogous to $\sim$ for the modified version of (35) where $g_1^{ok}()$ takes the place of $g_1()$. This can be summarised as $z_1 \sim_k z_2 \Rightarrow z_1 \sim z_2$ thus the equivalence classes of $\sim_k$ are nested within those of $\sim$. So any solution of $f_{sk1}(z) = f_{sk1}(g_1^{ok}(z))$ for $f_{sk1}()$ is a function of the equivalence classes of $\sim_k$ so $f_{s1}(z) = h(f_{sk1}(z))$ for some right-unique function $h()$.

From (29), $f()^{o(n-1)}$ can be applied giving $f^{on}(z) = f^{on}(g_1(z))$ so if $f()$ satisfies (29) so does $f^{on}()$ with the same $g_1()$ and any $n \in \mathbb{N}$. This also follows from Theorem (7.8).

The extension to multiple simultaneous equations of type (29) will depend on the analogous existence of simultaneous equivalence classes.

7.1 solutions of (30)

From here up to Theorem 7.13 may be unnecessary!

Lemma 6.4 is quite confusing because the concept of "solution" is being used in different contexts and the same equation (30) is being used in two different ways, one to determine the set of functions $f()$ satisfying this for a given $g_2()$, and the other to determine the set of singular points that are common to all members of this set. If $P = (z_1, w_1)$ is not such a point, even if it satisfies $w = f(z)$ and $f()$ as a function satisfies (30), then it is not necessarily true that P as a point satisfies (30) and because $w_1 \neq g_2(w_1)$ there is no reason for P to be a singular point unless $f'(z_1) = 0$ or ∞. An example of the latter is a singular point at $z = z_1$ if $f^*(z) = f(z_1 + (z - z_1)^2)$ which also satisfies (30) showing that $(z_1, f^*(z_1)) = (z_1, f(z_1))$ is a singular point of $f^*()$. Here $f^*()$ is taking the place of $f()$. In (30) let $z = h(t)$ for any $h() \in \mathcal{A}$ and define the function $f^*(t) = f(h(t))$. Then $f^*(t) = g_2(f^*(t))$ i.e.

Lemma 7.11. *If $f(z)$ satisfies* (30) *with $g_2()$ then so does $f(h(z))$ for any $h() \in \mathcal{A}$.*

This is a very general solution of (30) with the same $f()$ and $g_2()$ because the function $h() \in \mathcal{A}$ is general. However the general form of the solution of (30) with fixed $g_2() \in \mathcal{A}$ is not necessarily $f(z) = f_s(h(z))$ where $h() \in \mathcal{A}$ for any single fixed function $f_s()$ to be determined because it is conceivable that there could be more than one such function $f_s()$ and in fact Lemma 7.11 only shows that the set of solutions of (30) for fixed $g_2()$ is in general of the form

$$\bigcup_{f \in S} \left\{ \bigcup_{h() \in \mathcal{A}} f(h()) \right\}. \tag{40}$$

where S is a set of functions in $\mathcal{A}$ such that for any pair of functions $f_1()$ and $f_2()$ in S it is not the case that either

$$f_1(z) = f_2(h(z)) \text{ or } f_2(z) = f_1(h(z)) \tag{41}$$

for any $h() \in \mathcal{A}$. The point of the last condition is to eliminate duplication that would otherwise occur in the set defined by (40).

If the search is for the subset of left-unique solutions $f()$ of (30), this is likely not the most general solution for fixed $g_2()$ because, by Lemma 6.13, if $f()$ is left-unique and has a singular point then $f^*(z) = f(h(z))$ will have a singular point at the corresponding location regardless of $h()$. Moreover, $f_1()$ and $f_2()$ are left-unique in (41) and $f_2^{o-1}(f_2(z)) = z$ so the first part of (41) implies $h(z) = f_2^{o-1}(f_1(z))$, so the set S (now called S^*) is a set of functions such that no pair of them $f_1()$ and $f_2()$ is such that $h(z) = f_2^{o-1}(f_1(z))$ for any $h() \in \mathcal{A}$. This is clearly impossible if S^* has more than one element because $f_2^{o-1}(f_1(z)) \in \mathcal{A}$. Therefore the general solution of (30) for left-unique functions $f()$ takes the form

$$\bigcup_{h() \in \mathcal{A}} f^*(h()). \tag{42}$$

for some particular function $f^*() \in \mathcal{A}$ that must also be left-unique because $f()$ is by Lemma 7.5 assuming $h()$ is right-unique. The function $f^*()$ however is not unique because (42) can be written in many different ways using different functions $f^*()$, but in each equivalent way of writing the set of solutions there is only one function $f^*()$. Therefore the general form is $f(z) = f_s(h(z))$ where $f_s()$ is any of the special solutions of (30) that have the minimum number of singular points i.e. singular points only where $w = g_2(w)$ and $w = f(z)$ as required by Lemma 6.4 i.e. the function $f^*()$ above can be identified with $f_s()$. These arguments show that

This is very like Theorem 7.13.

Theorem 7.12. *A function $f() \in \mathcal{A}$ is left-unique and satisfies* (30) *with $g_2() \in \mathcal{A}$ if and only if $f(z) = f_{s2}(h(z))$ for some right-unique function $h() \in \mathcal{A}$ where $f_{s2}()$ is a left-unique solution of* (30) *with $g_2()$ that has the minimum number of singular points i.e. singular points (z, w) only where $w = f(z)$ satisfies $w = g_2(w)$.*

Requiring $f()$ in Theorem 7.12 to be left-unique eliminates for example $f(z) = z^2$ or $(\ln(z))^2$ so $(\ln(h(z)))^2$ is eliminated as a general solution of (30) by the left-uniqueness condition on $f()$. The general solution $f(z) = h(z)^2$ for arbitrary $h()$ is vacuous because it is equivalent to $f(z)^{1/2} = \pm h(z)$ which is the set of functions each of which is the union of a member of $\mathcal{A}$ and its negative and this must be true whatever $f()$ is and includes any function $f() \in \mathcal{A}$. Generally, the set of functions $f(z) = k(h(z))$ where $k()$ is right-unique and $h()$ is arbitrary implies $k^{o-1}(f(z)) = k^{o-1}(k(h(z)))$ which is a union that includes $h()$ itself and is arbitrary i.e. vacuous because if $h()$ is chosen as $k^{o-1}(f(z))$ then $k(h(z)) = k(k^{o-1}(f(z))) = f(z)$.

For $f(z) = (\ln(h(z)))^2$ for arbitrary $h()$, $f(z)^{1/2} = \pm \ln(h(z))$ so $f(z)^{1/2} = f(z)^{1/2} \pm 2\pi i$ which is not vacuous and means that for any value of $f(z)^{1/2}$, $2\pi i$ can be added or subtracted from it, and the sign can be changed to give another value of it. It can be written as $f(z) = (f(z)^{1/2} + 2\pi i)^2$ which is also an equation of the type (30) with general solution $f(z) = (\ln(h(z)))^2$ which should be part of a more general result than Theorem 7.12.

If $f()$ does not have to be left-unique, how can duplication be avoided in the general solution?

[This set obviously has a vast amount of repetition as defined by (40). Any $f^*() \in \mathcal{A}$ that is also of the form $f(h())$ for some other $f() \in \mathcal{A}$ should be excluded from the outer union in (40). Also because $f_s(h())$ are all solutions, the outer union must include the special solutions $f_s()$. Then the question is are there any other solutions? Let $f^*()$ satisfy (30) but not be one of $f_s(h())$. Then by Lemma 6.4 $f^*()$ as well $f_s()$ has singular points at every point (z, w) that is a solution of $w = g_2(w)$ where $w = f(z)$. Each subset of solutions for fixed $f()$. Take any solution $f()$ and identify singular points that are not required by Lemma 6.4 and use this to construct $h()$ or an equation it satisfies so we know it exists. see the dual result half way down page 33.]

[Equation (30) can be written as

$$z_1 = z_2 \Rightarrow f(z_1) = g_2(f(z_2)). \tag{43}$$

It is hypothesised that the special solutions also satisfy

$$z_1 = z_2 \Leftarrow f(z_1) = g_2(f(z_2)). \tag{44}$$

how could this ever be true?

i.e. they satisfy

$$z_1 = z_2 \Leftrightarrow f(z_1) = g_2(f(z_2)). \tag{45}$$

i.e. the equivalence relation defined by $z_1 \sim z_2 \Leftrightarrow f(z_1) = g_2(f(z_2))$ is just equality. Can we show that the s.p. of $f()$ satisfying this are included in those for any solution of (30)? About the simplest example possible is $f(z) = -f(z)$ having the equivalence relation $z_1 \sim z_2 \Leftrightarrow f(z_1) = \pm f(z_2)$ ($+$ sign for n even and $-$ sign for n odd). This is equivalent to $f^2(z_1) = f^2(z_2)$ so the equivalence relation is equality if $f^2()$ is left-unique. For example $f^2(z) = \ln(z)$ so $f(z) = \ln(z)^{1/2}$. The general solution is $h(z)^{1/2}$ and the special solutions are $f(z) = \left(\frac{a+bz}{c+dz}\right)^{1/2}$ where $h()$ left-unique and right-unique. If $f()$ is not left-unique it must satisfy (29) which has singular points determined in a manner similar to solutions of (30).

Theorem 7.13. *A function $f() \in \mathcal{A}$ satisfies (30) with $g_2() \in \mathcal{A}$ if and only if $f(z) = f_{s2}(h(z))$ for some function $h() \in \mathcal{A}$ where $f_{s2}()$ is a left-unique solution of (30) with $g_2()$, such that in addition, $f_{s2}()$ has the minimum number of singular points i.e. singular points $(z, w = f(z))$ only where w satisfies $w = g_2(w)$. If $f()$ is left-unique so is $h()$.*

Proof. Invert (30) to obtain $f^{o-1}(z) = f^{o-1}(g_2^{o-1}(z))$. This is (29) with $f()$ renamed as $f^{o-1}()$, and $g_1()$ renamed as $f_1^{o-1}()$. Apply these name changes to Theorem 7.8. □

Theorems (7.8) and (7.13) can be combined as follows because the general form which specialises $h(f_{s1}(z))$ and $f_{s2}(h(z))$, both by choosing $h()$ is $f_{s2}(h(f_{s1}(z)))$.

Theorem 7.14. *A function $f() \in \mathcal{A}$ satisfies (29) with $g_1() \in \mathcal{A}$, and (30) with $g_2() \in \mathcal{A}$ if and only if $f(z) = f_{s2}(h(f_{s1}(z)))$ for some function $h() \in \mathcal{A}$ where $f_{s1}()$ is a right-unique solution of (29) with $g_1()$, such that $f_{s1}()$ has the minimum number of singular points i.e. singular points $(z, w = f(z))$ only where z satisfies $z = g_1(z)$ and $f_{s2}()$ is a left-unique solution of (30) with $g_2()$ such that $f_{s2}()$ has the minimum number of singular points i.e. singular points $(z, w = f(z))$ only where w satisfies $w = g_2(w)$.*

This makes most sense if $h()$ is left-unique and right-unique i.e. a bilinear function. This is the simplest solution. In this case, $f_{s2}(h())$ and $h(f_{s1}())$ can be written as $f_{s2}()$ and $f_{s1}()$ respectively because of the arbitrariness in $f_{s1}()$ and $f_{s2}()$, so that $f(z) = f_{s2}(f_{s1}())$.

With reference to Theorems (7.8) and (7.13), (30) is satisfied by $f(z) = h(f_{s1}(z))$ so $f_{s2}(h(z)) = g_2(f_{s1}(h(z))$ for arbitrary $h()$ which can be called w therefore

$$f_{s2}(w) = g_2(f_{s2}(w)). \tag{46}$$

Also (29) is satisfied by $f(z) = h(f_{s1}(z))$ so $h(f_{s1}(z) = h(f_{s1}(g_1(z)))$ for arbitrary $h()$ therefore

$$f_{s1}(z) = f_{s1}(g_1(z)). \tag{47}$$

The operators S_1 and S_2 will be introduced here such that $f_{s1}() = S_1[g_1()]$ and $f_{s2}() = S_2[g_2()]$. Now it can be checked whether

$$f(z) = f_{s2}(h(f_{s1}(z)) \tag{48}$$

satisfies $f(z) = g_2(f(g_1(z)))$. Substituting this in gives $f_{s2}h(f_{s1}(z))) = g_2(f_{s2}(h(f_{s1}(g_1(z))))$ which simplifies to $f_{s2}(h(f_{s1}(g_1(z)))$ by (46) and to $f_{s2}(h(f_{s1}(z)))$ by (47) which is an identity, so it follows that $f_{s2}(h(f_{s1}(z))$ is indeed a very general (if not the most general) solution of $f(z) = g_2(f(g_1(z)))$.

Likewise in reverse, starting with $f_{s2}(h(f_{s1}(z)))$ using (47) then (46) it is equal to $g_2(f_{s2}(h(f_{s1}(g_1(z))))$ and finally using (49) it becomes $f(z)$ thus (48) follows. This proves that (48) and (49) are equivalent provided the functions $f_{s1}()$ and $f_{s2}()$ are defined in terms of $g_1()$ and $g_2()$ as in (46) and (47).

Inverting (46) gives $f_{s2}^{o-1}(z) = f_{s2}^{o-1}(g_2^{o-1}(z))$. Also $f_{s2}() = S_2[g_2()] = (S_1[g_2^{o-1}()])^{o-1}$. From (47) by substituting $g_1^{o-1}(t)$ for z gives $f_{s1}(g_1^{o-1}()) = f_{s1}()$ i.e. $S_1[k()] = S_1[k^{o-1}()]$ for any $k() \in \mathcal{A}$. Therefore $S_2[g_2()] = (S_1[g_2()])^{o-1}$ which is the relationship between $S_1[]$ and $S_2[]$. Elsewhere $S_1[]$ will simply be referred to as $S[]$.

Theorem 7.15. *The general solution of* (49) *for* $f()$ *is* $f(z) = f_{s2}(h(f_{s1}(z)))$ *i.e.* $f(z) = (S[g_2])^{o-1}(h(S[g_1](z)))$ *where the conditions on* $f_{s1}()$ *and* $f_{s2}()$ *and* $h()$ *in Theorems* (7.8) *and* (7.13) *apply and the operator* $S[]$ *is defined such that* $S[g_1()]$ *is the solution* $k()$ *of* $k(z) = k(g_1(z))$ *unique up to a bilinear function having singular points only where* $z = g_1(z)$.

How can $f_{s1}()$, $f_{s2}()$ be obtained practically from $g_1()$ and $g_2()$ respectively. What properties do these relationships $S_1[]$ and $S_2[]$ have? Do $f()$,$g_1()$ and $g_2()$ have to be single-component functions?

Suppose in addition to $f() \in \mathcal{A}$ satisfying (30) with $g_2()$ it satisfies (29) with $g_1()$ then $f()$ satisfies

$$f(z) = g_2(f(g_1(z))). \tag{49}$$

It is a strange thing that in lemmas (7.3) and (7.7), the defining conditions for the sets of common singular points of functions $f() \in \mathcal{A}$ satisfying (30) and (29) look like rewritten versions of the same equations. These equations have to be interpreted as equations for single-valued quantities to get these results.

Because of lemma 6.4, every solution of (29) has singular points at points z where $z = g_1(z)$ the type of which is determined as in Section 9 or extensions of it. Thus it is unnecessary to think about the types of the singular points of $h(f_{s1}(z))$.

For example $(f_{s1}(z))^{-2}$ which also has a singular point where $f_{s1}(z) = 0$. The uniqueness of the set of equivalence classes on which $f_{s1}()$ is based suggests that of all the solutions of (29) in $\mathcal{A}$, the *special* solutions $f_{s1}()$ that also satisfy (35) have singular points only where $z = g_1(z)$ because if there was another singular point in a special solution of (29) where $z \neq g_1(z)$ then all functions of the form $f(z) = h(f_{s1}(z))$ would by Lemma 6.6 also have a singular point there contradicting the above.

All the types of singular point so far found are of the types $q : p$ representing the winding number ratio where p and q are positive integers have no common factors. These are all the types of singular points for algebraic functions. In the cases where p and q are finite, a singular point (z_0, w_0) is a point about which if a path is traced from the starting point back to itself q times in the z plane this corresponds to a path in the w plane described p times back to itself.

The most general form of equations such as (13), (16), (18), (22) and (26) that describe the behaviour in the neighbourhood of a singular point seems to be

$$f(z) = g_2(z, f(g_1(z))) \tag{50}$$

in which g_2 has direct z dependence in addition to its dependence on $f()$. The conditions for singular points required by (50) are

$$\begin{array}{l} z = g_1(z) \\ f = g_2(z, f) \end{array} . \tag{51}$$

The meaning of (50) where $g_1()$ is the identity function is that there is an associated singular point (z_0, w_0) which is the point about which if a path in the z plane is followed to its starting point and if the function value is followed continuously, the values of the function at each end of the path are related by (50). This is the case where $q = 1$. As will be shown, the singular point is also a point where the number of function values changes and w_0 is given by the different values of the function $w_0 = f(z_0)$ being equal. This can be used to determine (z_0, w_0).

There is another version of this to describe the situation where $p = 1$. In this case $g_2()$ is the identity function and the roles of z and $w = f(z)$ are reversed. There is then a point (z_0, w_0) about which if a continuous path is traced in the w plane back to itself then the corresponding values of z are related by (50). The equality of these values determines the value z_0.

In addition to these cases, for non-algebraic functions it is possible to have $q = \infty$. In this case the value of w is never returned to its original value. Probably the simplest example is $w = f(z) = \ln(z)$ the inverse of the complex exponential function. This is equivalent to $z = \exp(w) = \exp(w).\exp(2\pi i) = \exp(w + 2\pi i)$. Therefore $w + 2\pi i = \ln(z)$ and equation (50) is satisfied for $f() = \ln()$ and $g_1(z) = z + 2\pi i$ and $g_2(z, f) = f$. Therefore the singular points are given by $z = z + 2\pi i$ from lemma 6.4 which implies $z = \infty$. This resolves the paradoxical situation with Theorem 1.3 and Lemma 4.2 and the fact that the exponential function has no finite singular points (they are at $z = \infty$ with $w = 0$ and ∞). As in the examples above $g_2()$ and $g_1()$ are right-unique and the singular point of $f()$ is at $z = 0$.

Analysis of behaviour in the neighbourhood of singular points similar to the above can be found for functions of a complex variable that are not algebraic as the following examples show.

Returning to $f(z) = \ln(z)$, it satisfies $f(z) = f(z) + 2\pi i$. Conversly $f(z) = f(z) + 2\pi i$ implies, taking the exp of both sides, the identity $\exp(f(z)) = \exp(f(z) + 2\pi i) = h(z)$ say, for some function $h(z)$ in $\mathcal{A}$ which is completely arbitrary because this imposes no condition on $h()$, therefore in general $f(z) = \ln(h(z))$. The singular point(s) of $f()$ are only where $h(z) = 0$ or ∞ and at points z that are singular points of $h()$. At minimum there are singular points of $f()$ only where $h(z) = 0$ or ∞ when $h(z) = a+bz$ so that $h()$ has no singular points. This implies $z_0 = -a/b$ or ∞ and the only fixed singular point is at $z_0 = \infty$ with the other one having an arbitrary location, and the singular point by 6.4 has w_0 given by the solution of the single-value equation $w_0 = w_0 + 2\pi i$ which is $w_0 = \infty$. Therefore the singular points of $\ln()$ are at $(0, \infty)$ and (∞, ∞) and those of its inverse $\exp()$ are at $(\infty, 0)$ and (∞, ∞).

Consider $w = (\ln(z))^2$. Can a similar analysis for this be done? We have $w = (\ln(z) + 2\pi i)^2$ then (50) is satisfied with $g_2(z, f) = (f^{1/2} + 2\pi i)^2$ and $g_1(s) = s$. Note that $g_2()$ is now not right-unique. Another analysis of this sort comes from $(\ln(z))^2 = (-\ln(z))^2 = (\ln(z^{-1}))^2$ i.e. Equation (50) with $g_2(z, f) = f$ and $g_1(z) = z^{-1}$, which shows that if in equation (50) either of $g_2()$ or $g_1()$ is not right-unique, this analysis may not be unique.

Consider $f(z) = z\ln(z)$, then

$$f(z) = f(z) + 2\pi iz. \tag{52}$$

This can be represented in terms similar to (50) with single valued $g_2()$ and $g_1()$ but this time g_2 has direct z dependence in addition to its dependence on $f()$ and $g_2(z, f) = 2\pi iz + f$ and $g_1(z) = z$. Conversely from (52), dividing by z and taking the exponential gives the tautology $\exp(f(z)/z) = \exp(f(z)/z)$, therefore this function can be any function in $\mathcal{A}$ say $h(z)$. Therefore $f(z)/z = \ln(h(z))$ and $f(z) = z\ln(h(z))$. The singular points of $f()$ are at any point where $h(z) = 0$ or ∞ or at any point that is a singular point of $h()$. This gives at minimum, where $h(z) = a + bz$ with $b \neq 0$, singular points at $z = -a/b$ and $z = \infty$. If it doesn't have this property how can it be transformed to a function that does have it, (52) being one example? Typically, if (29) is satisfied for two separate functions $g_1()$ or a multivalued $g_1()$ then $f()$ is expected to be constant because applying $g_1()$ and $g_2()$ and their inverses repeatedly to a z value will likely reach any value eg $f(z) = f(z^{1/2})$. What are the conditions for an exception to this.

Consider the binary operation which is the *special common solution* $f()$ of $f(z) = h_1(f_1(z))$ and $f(z) = h_2(f_2(z))$ for fixed $f_1()$ and $f_2()$, but arbitrary $h_1()$ and $h_2()$.

In order to generate $g_1()$ and $g_2()$ from $f()$ according to (29) and (30) this assumes that $g_1()$ and $g_2()$ are not dependent on z. In the general case this is not so as (52) shows. Generalising (52) to $f(z) = f(z)+k(z)$ for some $k() \in \mathcal{A}$. To solve this substitute $f(z) = f^*(z)k(z)$. Then it cancels down to $f^*(z) = f^*(z)+1$ so $2\pi i f^*(z) = 2\pi i f^*(z)+2\pi i$ and $\exp(2\pi i f^*(z))$ can be anything, call it $h() \in \mathcal{A}$, then $f^*(z) = \frac{\ln(h(z))}{2\pi i}$ and finally $f(z) = k(z)\frac{\ln(h(z))}{2\pi i}$ provided $k(z)$ is not the zero function $k(z) = 0$. From this the solution of $f(z) = k_1(z)f(z)$ can be obtained as $f(z) = k_1(z)^{\frac{\ln(h(z))}{2\pi i}}$ by a $\ln()$ transformation.

If $f(z) = k_1(z)f(z) + k_2(z)$ again make the substitution $f(z) = f^*(z)k_2(z)$ then it simplifies to $f^*(z) = k_1(z)f^*(z)+1$ and so $\exp(2\pi i f^*(z)) = \exp(2\pi i k_1(z)f^*(z))$ it may not be useful to go further.

Inversion may be useful too for example (52) and its solution can be written as

$f^{o-1}(z) =$ solution for w of $z = f(w) + 2\pi i w$. This has solution for $f()$ given by

$f^{o-1}(z) =$ solution for w of $w\ln(h(w)) = z$. This can be written as follows where $l() = f^{o-1}()$: $l(z) =$ solution for w of $z = l^{o-1}(w) + 2\pi i w$ has solution $l(z) =$ solution for w of $w\ln(h(w)) = z$. Better notation needed!

Consider $l(f(z)) = l(k(z)f(z))$ as a special case of (50).

What happens if (50) is inverted?

It seems paradoxical to say that $z\ln(h(z))$ is the general solution of (52) because (52) just states that whatever the multivalued function $f(z)$ is, if it has any value w at some point z, then at that point it also has the values $w+2\pi inz$ for all $n \in \mathbb{Z}$. In fact $z\ln(h(z))$ can be any function $f()$ in $\mathcal{A}$ provided $h(z) = \exp(\frac{f(z)}{z})$ and (52) holds in the multivalued sense. Nevertheless the use of the term "general solution" in this and other cases does seem convenient.

Suppose $f(z) = (\ln(z))^k$. Introduce the auxiliary function $g_1(z) = z^p$ then $f(g_1(z)) = (\ln(z^p))^k = p^k f(z)$ so (50) holds with $g_2(z, f) = fp^{-k}$, and Lemma 5.2 characterises $g_1()$. Alternatively, if only $f(g_1(z)) = p^k f(z)$ and

$g_1(z) = g_1(e^{2\pi i/p}z)$ then this is a set of defining equations for $f()$ involving two instances of (50) and linear functions only, one to characterise $g_1()$ and the other to define $f()$.

8 Compositional powers of a function

This means expressions of the form $f()^{on}$ where n can be a positive or negative integer that have been introduced earlier in the context of relations in general. There is a natural extension of this to other exponents such as any rational number or any value in $\mathbb{C}$. One neat way to do this is from a nice trick I found online [5] and is as follows. Introduce the function $f^*(t) = f^{ot}(z_0)$ where z_0 is an arbitrary value. Then $f^*()$ satisfies the following

$$f^*(t+1) = f^{o(t+1)}(z_0) = f(f^{ot}(z_0)) = f(f^*(t)) \tag{53}$$

then in terms of $f^*()$ a compositional power of $f()$ can be expressed as follows $f^{on}(f^*(t)) = f^{on}(f^{ot}(z_0)) = f^{o(n+t)}(z_0) = f^*(n+t)$ i.e.

$$f^{on}(w) = f^*(n + f^{*(o-1)}(w)). \tag{54}$$

Also (53) is $f^*(t) = f^{o-1}(f^*(t+1))$ has a formal solution as above given by $f^*(t) = S_2[f^{o-1}](h(S_1[z \to z+1](t)))$ from Theorem 7.15 so $f^{on}(w) = S_2[f^{o-1}](h(S_1[z \to z+1](n + [S_2[f^{o-1}](h(S_1[z \to z+1](t))]^{o-1}(w))$.

Example: $f(f(z)) = z^2$. The solution should be $f(z) = z^{(2^{1/2})}$ but because $2^{1/2}$ is irrational, $f(z)$ has an infinite number of values!

9 The relationship between $g_1()$ and the type of singular points of $f()$ satisfying (29)

Consider the role played by $g_1()$ and its derivatives at an intersection point z_1 which is a solution of $g_1(z) = z$. This as will be seen controls to leading order the behaviour of $f(z)$ in the neighbourhood of the singular point at z_1 provided $f(z)$ satisfies (29) where $g_1()$ is as in (29). First consider an arbitrary value of $g_1'(z_1)$. For $z \approx z_1$, $g_1(z) \approx g_1(z_1) + (z - z_1)g_1'(z_1) = z_1 + (z - z_1)g_1'(z_1)$ therefore $f(z) \approx f(z_1 + (z - z_1)g_2'(z))$. Put $z = z_1 + \delta$ and treating this as an equality then $f(z_1 + \delta) = f(z_1 + \delta g_1'(z_1))$. A change of variable can now be made so as to relate this equation to $f(z) = f(z) + 2\pi i$ with its known solution. Let $w = \ln(\delta) = \ln(z - z_1)$ and the new function $f^*()$ by $f^*(w) = f(z)$ then $f^*(w) = f^*(w + \ln g_1'(z_1))$. Now let $w = \alpha t$ and $f^+(t) = f^*(w) = f(z)$ then $f^+(t) = f^+\left(t + \frac{\ln g_1'(z_1)}{\alpha}\right)$. Then choose α so that $\ln(g_1'(z_1)/\alpha = 2\pi i$ i.e.

$\alpha = \frac{\ln(g_1'(z_1)}{2\pi i}$ then $f^+(t) = h(\exp(t))$ i.e.

$$f(z) = f^*(w) = h(\exp(w/\alpha)) = h\left((z - z_1)^{\frac{2\pi i}{\ln(g_1'(z_1))}}\right). \tag{55}$$

This is the asymptotic behaviour of $f()$ for z close to z_1 where $h()$ is an arbitrary function in $\mathcal{A}$. This works provided $g_1'(z_1) \neq 0$.

Now suppose $g_1'(z_1) = 0$ but $g_1''(z_1) \neq 0$. Then $g_1(z) \approx g_1(z_1) + \frac{(z-z_1)^2}{2} g_1''(z_1)$ then $f()$ satisfies $f(z) = f\left(z_1 + \frac{(z-z_1)^2}{2} g_1''(z_1)\right)$. Now put $k(\delta) = f(z_1 + \delta)$ where as before $\delta = z - z_1$ then $k(\delta) = k(\delta^2 g_1''(z_1)/2)$. Introduce $k^*()$ by $k(\delta) = k^*(\ln(\delta))$ then $k^*(\ln(\delta)) = k^*(2\ln\delta + \ln(g_1''(z_1)) - \ln(2))$. Introduce w by $w = \ln\delta$ then $k^*(w) \approx k^*(2w)$ because as $\delta \to 0$, $|w| \to \infty$ so the other terms can be asymptotically ignored. Now introduce $k^+()$ by $k^+(\ln(x)) = k^*(x)$ then $k^+(\ln w) = k^+(\ln w + \ln 2)$ so $k^+(u) = k^+(u + \ln 2)$ where $u = \ln w$. Now let $t()$ be defined by $t(u\beta) = k^+(u)$ then $t(u\beta) = t(u\beta + \beta\ln(2))$. Choosing β to be $\beta = \frac{2\pi i}{\ln(2)}$ then $t(x) = t(x + 2\pi i)$ from which $t(x) = h(\exp(x))$. Undoing all these transformations now shows that $t(x) = t(u\beta) = k^+(u) = k^+(\ln(w)) = k^*(w) = k(\delta) = f(z_1 + \delta) = f(z)$ and $h(\exp(x)) = h(\exp(\beta u)) = h(\exp(\beta\ln(w))) = h(w^\beta) = h([\ln(z - z_1)]^\beta)$ so finally

$$f(z) = h\left([\ln(z - z_1)]^{\frac{2\pi i}{\ln 2}}\right) \tag{56}$$

where this result will only be asyptotically correct as $z \to z_1$. Note that $g_1''(z_1)$ is not involved.

From (55) $g_1'(z_1) = 1$ is obviously also a special case needing separate treatment. Then $g_1(z) \approx z + \frac{(z-z_1)^2}{2} g_1''(z_1)$ and the equation to be solved is $f(z) = f\left(z + \frac{(z-z_1)^2}{2} g_1''(z_1)\right)$. Putting $z = z_1 + \delta$ and introducing $f^*(\delta) = f(z_1 + \delta)$ gives

$$f^*(\delta) = f^*\left(\delta + \frac{\delta^2}{2} g_1''(z_1)\right). \tag{57}$$

Introduce the new variable k by $k\left(\delta + \frac{\delta^2}{2} g_1''(z_1)\right) - k(\delta) = \Delta$ so that the iteration of (57) is transformed to an arithmetic progression, then for small δ, $\frac{\delta^2}{2} g_1''(z_1) k'(\delta) = \Delta$ which can be integrated and inverted to give $\delta = -\frac{2\Delta}{k g_1''(z_1)}$. Then $f^*\left(\frac{-2\Delta}{k g_1''(z_1)}\right) = f^*\left(\frac{-2\Delta}{k g_1''(z_1)} + \frac{2\Delta^2}{k^2 g_1''(z_1)}\right)$. Introducing $f^+(k) = f^*(\delta)$ this can be written in terms of $f^+()$ as $f^+(k) = f^+\left(\frac{\frac{-2\Delta}{g_1''(z_1)}}{\left(\frac{-2\Delta}{k g_1''(z_1)} + \frac{2\Delta^2}{k^2 g_1''(z_1)}\right)}\right)$ which simplifies to $f^+(k) = f^+\left(\frac{k^2}{k-\Delta}\right) \approx f^+(k + \Delta)$. Let $g()$ be given by $g(l) = f^+(k)$ where $k = l/\alpha$ then $g(l) = g(l + \alpha\Delta)$ and choosing $\alpha\Delta = 2\pi i$ then $g(l) = h(\exp(l))$

where $h()$ is arbitrary and this implies

$$f(z) = h\left(\exp\left(-\frac{4\pi i}{g_1''(z_1)(z-z_1)}\right)\right) \tag{58}$$

asymptotically as $z \to z_1$.

This result can be generalised as follows. Suppose $g_1'(z_1) = 1$ and $g_1^{(n)}(z_1) = 0$ for $2 \leq n \leq m-1$ and $g_1^{(m)}(z_1) \neq 0$ for $m \geq 2$. Then $g_1(z) = z + \frac{(z-z_1)^m}{m!} g_1^{(m)}(z_1) + O(z-z_1)^{m+1}$. In terms of $f^*()$ and δ as above, $f(z) = f(g_1(z))$ becomes $f^*(\delta) = f^*\left(\delta + \frac{\delta^m g_1^{(m)}(z_1)}{m!} + O(\delta^{m+1})\right)$. This can be iterated and if k is chosen such that $k\left(\delta + \frac{\delta^m g_1^{(m)}(z_1)}{m!}\right) = k(\delta) + \Delta$ which can be approximated by $k'(\delta)\frac{\delta^m g_1^{(m)}(z_1)}{m!} = \Delta$ which integrates to $k(\delta) = \frac{-\Delta m!}{(m-1)\delta^{m-1} g_1^{(m)}(z_1)}$, then the iteration is an arithmetic progression and $f^*(\delta) = f^+(k) = f^+(k+\Delta)$. Therefore similarly to the above,

$$f(z) = h\left(\exp\left(\frac{-2\pi i m!}{(m-1)g_1^{(m)}(z_1)(z-z_1)^{m-1}}\right)\right) \tag{59}$$

asymptotically as $z \to z_1$.

10 Some interesting examples

This one doesn't seem to make much sense! Another example is

$$f(z) = (f(z))^{1/2} \tag{60}$$

with a singular point where $f(z) = 0$, which is a special case of (79) in which $g_2()$ is not single valued. Taking natural logarithms twice gives

$$\ln\ln(f(z)) = \ln(1/2) + \ln\ln(f(z)) \tag{61}$$

and so

$$\frac{2\pi i}{\ln(2)}\ln\ln(f(z)) = -2\pi i + \frac{2\pi i}{\ln(2)}\ln\ln(f(z)) \tag{62}$$

so

$$\exp\left(\frac{2\pi i}{\ln(2)}\ln\ln(f(z))\right) \tag{63}$$

is arbitrary so call it $h(z)$ then

$$f(z) = \exp\left(\exp\left(\frac{\ln(2)}{2\pi i}\ln(h(z))\right)\right). \tag{64}$$

The function $f()$ can only have a singular or inversion point where $h()$ has singular or inversion point(s) or where $h(z) = 0$ or ∞ so $f(z) = 0$ or ∞. This log-like singularity from (60) is characterised by the equations

$$\begin{array}{l} g_2(z) = -g_2(z) \\ f(z) = g_2(f(z)) \end{array} \tag{65}$$

for the multivalued functions $f()$ and $g_2()$, where $g_2()$ is the *special solution*. If $f()$ is also the *special solution* then

$$f(z) = \exp\left(\exp\left(\frac{\ln(2)}{2\pi i}\ln(a+bz)\right)\right) \tag{66}$$

where a and b are constants.

Next consider

$$f(z) = f(z^2)/2. \tag{67}$$

This is a special case of (79) in which the condition for a singular point is more complicated than for (29) for which the condition for a singular point would give

$$z = g_1(z) = z^2 \tag{68}$$

determining more than one such point i.e. $z = 0, 1$. The effect of the extra factor of 2 complicates this a bit but this is still clearly true. Because $g_1()$ is not left-unique, (68) relates new singular points to other points already known to be singular points. In this example the singular points are dense on the unit circle because these are points for which $z^{(2^k)} = 1$ for arbitrarily large k. It follows that $f(z) = f(z^2)/2 = f(z^4)/4 = \ldots f(z^{(2^k)})/2^k$ so if $z = re^{i\theta}$, $f(re^{i\theta}) = f((re^{i\theta})^{2^k})/2^k$ for all $k > 0$. For fixed r and θ suppose $\theta + 2\pi p \approx 2^k\theta$ where $p, k \in \mathbb{N}$ then $f(r^{(2^k)}e^{i\theta}) \approx 2^k f(re^{i\theta})$. Putting $R = r^{(2^k)}$ gives $f(Re^{i\theta}) \approx \frac{\ln(R)}{\ln(r)} f(re^{i\theta})$. The log dependence on R behaviour at large R and the positions (z) of the singular points may suggest the following formula

$$f(z) = \int_0^{2\pi} d\theta \log_2 |z - e^{i\theta}|. \tag{69}$$

for a solution of (67). Its proof is as follows

$$\begin{array}{l} f(z^2) = \int_0^{2\pi} d\theta \log_2 |z^2 - e^{i\theta}| = \int_0^{2\pi} d\theta \log_2 \left(|z + e^{i\theta/2}||z - e^{i\theta/2}|\right) \\ = \int_0^{2\pi} d\theta \log_2 |z + e^{i\theta/2}| + \int_0^{2\pi} d\theta \log_2 |z - e^{i\theta/2}| \\ = 2\int_0^{\pi} d\theta \log_2 |z + e^{i\theta}| + 2\int_0^{\pi} d\theta \log_2 |z - e^{i\theta}| \\ = 2\int_{\pi}^{2\pi} d\theta \log_2 |z + e^{i(\theta-\pi)}| + " \\ = 2\int_{\pi}^{2\pi} d\theta \log_2 |z - e^{i\theta}| + " \\ = 2\int_0^{2\pi} d\theta \log_2 |z - e^{i\theta}| = 2f(z) \end{array} . \tag{70}$$

This example has really peculiar properties because $f(z)$ is ∞ on the unit circle and this appears to isolate the function into two regions that can behave somewhat independently because (68) is satisfied for $f()$ replaced by $af()$ where $a \in \mathbb{C}$ and clearly any two different values of a can be chosen inside and outside the unit circle and the solutions can be described as having a natural boundary on the unit circle. [This doesn't work for finite prescribed values because if finite values are prescribed on any closed contour the Cauchy integral formula determines a function that is everywhere analytic and finite, uniquely inside it, but does it work for the outside region?] This is an example that divides $\overline{\mathbb{C}}$ into two domains of holomorphy [4] that overlap only on the unit circle.

Next follows an intriguing example where the condition for a singular point (an equation of the type (29)) determines two of them and the solutions found satisfy an additional equation of the type (79). Suppose $g_1(z) = \frac{a+bz}{c+z}$. Then $g_1(z) = z$ is a quadratic equation with solutions say z_1 and z_2 such that $z_1 + z_2 = b - c$ and $z_1 z_2 = -a$ and $g_1(z)$ can be written as $g_1(z) = \frac{-z_1 z_2 + bz}{b - z_1 - z_2 + z}$. However in this case, $g_1()$ is left-unique and single valued so only two singular points arise as a result of (29) which becomes in this case

$$f(z) = f\left(\frac{bz - z_1 z_2}{b - z_1 - z_2 + z}\right). \tag{71}$$

Therefore by lemma 7.7 solutions of (71) have singular points at z_1 and z_2.

Using methods similar to those used in deriving (55) it possible to formally derive

$$f(z) = h_k\left(\sum_{n\in\mathbb{Z}} c_n \exp\left(\frac{2\pi i(\ln(z - z_k) + 2n_1\pi i)}{\ln\left(\frac{z_1-b}{z_2-b}\right) + 2n\pi i}\right)\right) \tag{72}$$

for $k = 1, 2$ where $h_1()$ and $h_2()$ are arbitrary functions. By trial and error, the following are possible solutions of (71):

$$f(z) = \left(c_n \frac{z - z_1}{z - z_2}\right)^s \tag{73}$$

where $s = \frac{2\pi i}{\ln\left(\frac{z_1-b}{z_2-b}\right)+2n\pi i}$ and $n \in \mathbb{Z}$. It is easy to show that

$$\frac{g_1(z) - z_1}{g_1(z) - z_2} = \frac{(z_1 - b)(z_1 - z)}{(z_2 - b)(z_2 - z)}. \tag{74}$$

Therefore

$$f(g_1(z)) = \left(c_n \frac{z - z_1}{z - z_2}\right)^s \left(\frac{b - z_1}{b - z_2}\right)^s. \tag{75}$$

The extra factor is $\left(\frac{b-z_1}{b-z_2}\right)^s$ can be written (including all its possible values) as

$$\exp(s\ln(t)) = \exp\left(\frac{\ln(t)\times 2\pi i}{\ln(t)+2n\pi i}\right) = \exp\left(\left(\frac{\ln(t)+2n_1\pi i}{\ln(t)+2n\pi i}\right)\times 2\pi i\right) = E_{n_1,n} \tag{76}$$

where $n_1, n \in \mathbb{Z}$ for some specific value of $\ln(t)$ and where $t = \frac{b-z_1}{b-z_2}$. Increasing n_1 by 1 adds $\frac{2\pi i\times 2\pi i}{\ln(t)+2n\pi i}$ to the argument of exp() multipling the whole expression by $\exp\left(\frac{-4\pi^2}{\ln(t)+2n\pi i}\right)$ and $E_{n,n} = 1$. From these it follows that $E_{n_1,n} = \exp\left(\frac{4\pi^2(n-n_1)}{\ln(t)+2n\pi i}\right)$. Therefore

$$f(g_1(z)) = \left(c_n\frac{z-z_1}{z-z_2}\right)^{\frac{2\pi i}{\ln\left(\frac{z_1-b}{z_2-b}\right)+2n\pi i}} \exp\left(\frac{4\pi^2(n-n_1)}{\ln\left(\frac{b-z_1}{b-z_2}\right)+2n\pi i}\right). \tag{77}$$

From (29)

$$f(z) = \exp\left(\frac{2\pi i}{\ln(t)+2n\pi i}\times\ln\left(c_n\frac{z-z_1}{z-z_2}\right)\right). \tag{78}$$

Taking this continuously round a small circuit C_1 anticlockwise round z_1 given by $z = z_1+\epsilon e^{i\theta}$ for $0\le\theta\le 2\pi$ where ϵ is a very small positive real number gives $f(z) = \exp\left(\frac{2\pi i}{\ln(t)+2n\pi i}\ln\left(\frac{c_n e^{i\theta}}{z-z_2}\right)\right) = \exp\left((\ln(c_n)+i\theta-\ln(z-z_2))\frac{2\pi i}{\ln(t)+2n\pi i}\right)$ The difference over the path C_1 of the argument of exp() is $\frac{2\pi i.2\pi i}{\ln(t)+2n\pi i}$ so the factor associated with doing C_1 is $\exp\left(\frac{-4\pi^2}{\ln(t)+2n\pi i}\right)$ i.e. $f(z)$ satisfies $f(z) = f(z)\exp(\frac{-4\pi^2}{\ln(t)+2n\pi i})$. This can be applied to write (77) as (73) verifying the assumed form of $f()$ though this is probably not its most general form. Doing the same thing for a small circuit C_2 anticlockwise round z_2 gives the equivalent result $f(z) = f(z)\exp(\frac{4\pi^2}{\ln(t)+2n\pi i})$.

11 Special solutions of the equations defining singular points

******* This section seems as if there are some very important results to be found but it needs quite a lot of work yet **********

*'s indicate likely theorems that have not yet been proved.

Let the binary relation $\succ$ on functions in $\mathcal{A}$ be defined by
$f()\succ g() \Leftrightarrow$ there exists an function $h()$ in $\mathcal{A}$ such that $f() = h(g())$. Then the relation $\succ$ that points towards the simpler function is reflexive and transitive. Also

Theorem 11.1. *If $f()\succ g()$ and $g()\succ f()$ then $f(z) = \frac{a+bg(z)}{c+dg(z)}$ for some finite constants $a, b, c, d \in \mathbb{C}$.*

Proof. Suppose $f() \succ g()$ and $g() \succ f()$ then $f() = h_1(g())$ and $g() = h_2(f())$ for some functions $h_1()$ and $h_2()$ in $\mathcal{A}$, and therefore $f() = h_1(h_2(f()))$ i.e. $h_1(h_2()) = I()$ which has no singular point. By Theorem 82 $h_1()$ can have no singular point and is therefore a bilinear function and so $f(z) = \frac{a+bg(z)}{c+dg(z)}$. □

If $f() \in \mathcal{A}$ is left-unique and right-unique then $f()^{o-1}$ has the same properties . $f()$ and f^{o-1} are right-unique implies so are $f()of^{o-1}()$ and $f^{o-1}()of()$ and their inverses which are same are both left-unique so these are both the identity i.e. $f(f^{o-1}(z)) = f^{o-1}(f(z)) = z$. Because this has no singular point

$f(f^{o-1}(z)) = f^{o-1}(f(z)) = z$ **********Prove that a left-unique and right-unique function is bilinear *********** Suppose a set T of functions in $\mathcal{A}$ is such that if $f() \in T$ then $h(f()) \in T$. Then this set is determined by the set $R \subseteq T$, the root functions, such that for any function $f()$ in T there exists a member $g() \in R$ such that $f() \succ g()$. Such a set T will be called a rooted set, rooted by the set R. Suppose a single root function $r()$ acts a root for T i.e. $\forall f() \in T[f() \succ r()]$. Suppose another function $r_1()$ also has this property, then $\forall f() \in T[f() \succ r_1()]$ and in particular $r() \succ r_1()$. Likewise $r_1() \succ r()$.

Then by Theorem 11.1 $r(z) = \frac{a+br_1(z)}{c+dr_1(z)}$. Such a rooted set will be called singly-rooted. Thus the root functions associated with a singly rooted set are related by a bilinear transformation.

From Theorem 11.1 any root function $k()$ is unique up to a bilinear function or transformation (also known as a Möbius transformation or a linear fractional transformation) i.e $g_1(z) = \frac{a+bk(z)}{c+dk(z)}$ so a root function is actually a set of functions each member of which is related to any other member like this for some set of values $a, b, c, d \in \mathbb{C}$ such that $ad - bc \neq 0$. The terminology below will for simplicity refer to this special set just as a single function, the root function.

Lemma 11.2. *Every function in $\mathcal{A}$ is in the set rooted by a left-unique function.*

Proof. If $g()$ is left-unique then $f(z) = f(g^{o-1}(g(z)))$ so $f() \succ g()$. □

Theorem 11.3. *If $gof() \equiv f(g()) = g(f())$ and $g()$ is left-unique and right-unique then $f() \oplus g() = gof()$.*

Proof. The condition on $g()$ gives $g^{o-1}(g()) = I$ and $g(g^{o-1}()) = I$ and also $f(g()) = g(f())$. Suppose $l(z) = h_1(f(z))$ and $l(z) = h_2(g(z))$ for some arbitrary analytic functions $h_1()$ and $h_2()$. Then $f(z) = f(g^{o-1}(g(z))) = g^{o-1}(g(f(z))) = g^{o-1}(f(g(z)))$. Therefore $f(g^{o-1}(w)) = g^{o-1}(f(w))$ generally [where $w = g(z)$] and $l(z) = h_1(f(g^{o-1}(g(z))) = h_1(g^{o-1}(f(g(z))) = h_3(f(g(z))$ where $h_3() = h_1(g^{o-1}())$. According to the criterion for a root function, $f(g())$ is the required root function for the set of possible analytic functions $l()$. □

By considering the example when R_1 is $z \to z^2$, R_2 is $z \to z^3$ and I was expecting the intersection to be given by the root function $z \to z^6$ (R_3).

Returning to functions in $\mathcal{A}$ there is a kind of discreteness in them which is exemplified by the fact that there does not appear to be a function $f()$ such that $z^4 \succ f() \succ z^2$ and $z^2 \not\succ f()$ and $f() \not\succ z^4$.

11.1 old work

A common type of equation defining behaviour around a singular point is

$$f(z) = g_2(f(g_1(z))) \tag{79}$$

where $g_2()$ and $g_1()$ are right-unique functions. [what is it that makes $g_2(z) = z + 1$ and $g_1(z) = z/e$ a trivial or not useful example of (79) for $f(z) = \ln(z)$ whereas $g_2(z) = z + 2\pi i$ and $g_1(z) = z$ is not? it doesn't relate one branch to another and is interpretable ($f(z) = 1 + f(z/e)$) without considering $f()$ as multivalued. Is it that both $g_2()$ and $g_1()$ are not the identity? Try using two equations (29) and $f(z) = g_2(f(z))$. First look at algebraic functions.]

Here the equality is between two sets of values. The more general form

$$f(z) = g_2(z, f(g_1(z))) \tag{80}$$

occurs later. [the examples of this all came from combining solutions of (79) with z using arithmetic functions.]

Most of the examples above are actually special cases of

$$f(z) = g_2(z, f(z)). \tag{81}$$

****

A direct proof seems difficult. Suppose $z_1 \neq g_1(z_1)$. The condition for $f_s()$ to have no singular point at P, $(z_1, f_s(z_1))$, using (31) and (35), is that there is a neighbourhood N of P such that for all points $(z_2, f_s(z_2))$ and $(z_3, f_s(z_3)) \in N$, $z_2 = z_3 \Leftrightarrow z_2 \sim z_3$. This reduces to

$$\exists N \text{ of } P \left\{\forall (z_2, f_s(z_2)), (z_3, f_s(z_3)) \in N[\exists n \in \mathbb{N}[z_2 = g_1^{on}(z_3))] \Rightarrow z_2 = z_3]\right\}. \tag{82}$$

To establish this it is sufficient to choose N so small that if z_3 is included by being sufficiently close to z_1 that none of $g_1(z_3), g_1(g_1(z_3))$ etc. are included i.e. the images of N under $g_1()$ repeated any number of times must not overlap N itself.

The above arguments have assumed that $g_1()$ is single valued. It should be possible to do this with a multivalued $g_1()$ because then $z_1 = g(z_2)$ just means that z_1 is one of the values of $g(z_2)$. The equivalence classes are now more complicated to construct but the principle is the same.

Therefore these special fundamental solutions of (29) that also satisfy (35) will be called the *special* solutions of (29). The reason that the word "simplest" is no longer used is that this would be a constant function. Let $f_s^*()$ be another function that satisfies the conditions on $f_s()$ above then $f_s^*(z) = h^*(f_s(z))$ for some function $h^*()$ in $\mathcal{A}$. Also $f_s^*()$ has the same singular points as $f_s()$ (which is a requirement of a special solution of (29)), which by Lemma (6.6) is true if and only if $h^*()$ has no singular points i.e. by Theorem 82 $h^*()$ is a bilinear function. Therefore

[If $h^*()$ is any function without a singular point (i.e. a bilinear function by Theorem 82) then by Lemma (6.6) $f_s^*()$ will have singular points precisely where $f_s()$ does and $f_s^*()$ will also satisfy (35) i.e. the set of special solutions of (29) must include $\frac{a+bf_s(z)}{c+df_s(z)}$ if $f_s(z)$ is included for all $a, b, c, d \in \mathbb{C}$. Can there be any more? Any other such solution must take this form with a different function $h^*()$ that will be not bilinear and so must have at least two singular points somewhere and by Lemma 6.6, in accordance with the above argument, $h^*(f_s(z))$ must have extra singular points. This contradicts the above argument that $h^*(f_s(z))$ must have no singular or inversion point except when z satisfies $z = g_1(z)$.]

Theorem 11.4. *The set of special solutions to* (29) *i.e. those that also satisfy* (35) *have singular points only where* $z = g_1(z)$ *where* $g_1()$ *is as in* (29). *This set is the same as the set* $\frac{a+bf_s(z)}{c+df_s(z)}$ *for arbitrary* $a, b, c, d \in \overline{\mathbb{C}}$ *if* $f_s(z)$ *is itself a special solution of* (29). *Any solution to* (29) *can be written as* $h(f_s(z))$ *for some special solution* $f_s(z)$ *for some function* $h()$ *in* $\mathcal{A}$.

[perhaps there are important ideas here but needs working out! If $g_1()$ is not a linear function the equation $z = g_1(z)$ that determines the singular points could have many solutions, [and $g_1()$ itself could be described by another equation of the type (29) or (81) etc..] In such a case the original equation (29) for $f()$ together with other similar equations to determine $g_1()$ etc. could determine behaviour at a set of singular points simultaneously. In such a case it might be a good idea to try to solve for the singular points and then with $g_1()$ replaced by linear functions that give the same singular points, analyse each separately using the results in Section 9 or extensions if necessary, and then try to reconstruct the original function $f()$ but note example (67) indicating that in this case an infinite number of singular points can sometimes occur.]

Now consider iteration applied to (80) which gives

$$f(z) = g_2(z, g_2(g_1(z), f(g_1^{o2}(z)))) = ... = \\ g_2(z, g_2(g_1(z), g_2(g_1^{o2}(z), g_2(g_1^{o3}(z), g_2(g_1^{o4}, \ldots) \ldots) = [g_2((), f(g_1()))]^{on}(z) \tag{83}$$

where g_2 appears n times in this expression. Now suppose $g_1^{on}()$ is the identity function : $z \to z$ then

$$f(z) = g_2(z, g_2(g_1(z), g_2(g_1^{o2}(z), \ldots g_2(g_1^{o(n-1)}(z), f(z)) \ldots). \tag{84}$$

This is last expression depends independently on z and $f(z)$ through the functions $g_2()$ and $g_1()$ and can therefore be written as $k(z, f(z))$ i.e. (84) can be written in the form (81) for different $g_2()$ and $g_1()$. Also it is conceivable that (84) for some value of n takes the simpler form (29) again for different $g_2()$ and $g_1()$. In either of these cases the *special solution* of the respective iterated form of (80) can be defined as above. If this can be done for both cases the following example suggests this might define the *special solution* for (80) itself.

There are many results that can be obtained relating the solution sets of (79) with different values of $g_2()$ and $g_1()$. If (79) holds then the same relationship holds with $f()$ replaced by $k(f(l()))$, $g_2()$ replaced by $k(g_2(k^{o-1}()))$ and $g_1()$ replaced by $l^{o-1}(g_1(l()))$. Making these substitutions gives the same relationship with the function $k()$ applied to both sides and expressed in terms of the independent variable w given by $z = l(w)$. For example suppose $k(z) = az + b$ and $l(z) = cz + d$ then the function $f^*(z) = k(f(l(z))) = af(cz + d) + b$ satisfies $f^*(z) = g_1^*(f^*(g_2^*(z)))$ i.e. (79) with $g_1^*(z) = ag_2((z - b)/a) + b$ and $g_2^*(z) = (g_1(cz + d) - d)/c$.

If in equation (79) $g_2^{o-1}()$ is applied to both sides and the result expressed in terms of the variable $w = g_1(z)$ then the same relationship holds with $g_2()$ replaced by $g_2^{o-1}()$ and $g_1()$ replaced by $g_1^{o-1}()$.

The inverse functions of both sides of Equation (79) again give an equation of the same form showing that f^{o-1} satisfies the equation of the same form but with $g_2()$ replaced by $g_1^{o-1}()$ and $g_1()$ replaced by $g_2^{o-1}()$.

In these general arguments, it has to be borne in mind that $f^{o-1}(f(z))$ could have several components and is not necessarily just the identity function as in section 1.

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