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Partial differential equations

1 Introduction

The literature of basic work on partial differential equations (PDE's) and systems of them seems to be mainly divided into two parts, that based on knowledge of modern differential geometry and that just based on multivariable calculus. The first part I am not so familiar with and I am just getting to grips with what seems most important ([5, 6, 7, 8]) of which Boothby's book has been most useful for the basic concepts. This geometric approach is based on (1) regarding the independent and dependent variables all on the same footing using a coordinate-free methods and (2) using the basic ideas of differential geometry such as vector fields, forms, (types of tensors) and exterior algebra, which is very elegant but quite complicated. Much of this work was synthesised in the work of Élie Cartan. I think it likely that this geometric approach will provide another way of getting at essentially the same results found here.

However the older work is often quite hard to read due to difficult notation and concepts (apart from the annoying gothic script letters!). This is not often referred to in the basic texts. This document makes minimal use of this material except some of what is in Olver's book [2].

Firstly there is a heuristic argument showing a general equivalence to coupled sets of ODE's in different directions, but these directions depend in general on the boundary conditions of the original PDE problem. This argument suggests that for a set of p PDE's of first order involving p unknowns, p characteristic directions can always be found but some might coincide with each other or might involve complex numbers. The concept probably arose initially in the case of the wave equation in one variable of space (x) and one of time (t) $\partial^2\phi/\partial t^2 = c^2\partial^2\phi/\partial x^2$ and plays a central role in the well-known theory of linear second order PDE's with two independent variables and the appropriate type of boundary/initial conditions for a unique solution (see for example [4] or [3]).

It goes something like this. If a system of PDE's has solutions for a set of dependent variables say $u_1, u_2, \ldots u_p$ as functions of the independent variables $x_1, \ldots x_n$ in a region containing the initial given data and these solutions are assumed unique in a region as determined by the Cauchy-Kovalevskaya (CK) theorem, which is described in Olver [2] showing the uniqueness of the solution of the system in a neighbourhood of a surface where initial values of all the unknowns are specified but naturally this can only be done (section 2.6) if the initial surface does not contain any of the characteristic directions. Then if hypothetically all the dependent variables except one are given their values (by an oracle that somehow managed to guess them) the system would then be a system of first order PDE's for the single remaining unknown. Just one such PDE with the CK theorem would then determine a unique solution for this unknown using the initial data. These latter problems can always be solved for the remaining unknown by integrating along "strips" from an initial surface provided it does not contain any of these directions (Cauchy/Monge's method)[4]. This process could be repeated, updating each unknown in turn, and the whole cycle repeated until convergence starting with an initial estimate of all the unknowns consistent with the initial data. This argument suggests that in general there will be p directions to integrate to get the solution i.e. the original system is equivalent to a set p coupled systems of ODE's one for each unknown.

In the general case nothing can be said about these directions because they depend on the boundary conditions, but in many special cases of systems of PDE's some information about these directions is available from the original system itself. This is because there are simplifications that are independent of the boundary conditions can be searched for i.e. minimisation of dimension (independent and dependent variables). A reduced number of independent variables in which the equations can be expressed implies that any characteristic directions must be within the submanifolds defined by this reduced number of variables. Earlier I proposed that the numbers of independent and dependant variables be minimised and described methods for finding them [1]. In this paper I have developed this a little giving examples of what could be termed "partial" minimisation of dimension. This can give rise to some interesting cases such as when only two characteristic directions appear when three were expected showing that this is a case where two characteristic directions coincide but they are not necessarily in involution because if they were it would result in a reduction of dimension. Most treatments of this problem seem to not go much beyond identifying the characteristics and the relationship between these and the domains of influence and dependence of the solution on given data.

The main theme of the theory of PDE's seems to me to be to classify and characterise these special cases many of which are well-known and some of which I have identified here and in my earlier work[1].

When considering analytic systems of partial differential equations (PDE's) in general, two preliminary steps need to be taken first. (1) to simplify this to the treatment of a first order system, because any such system can be made first order by introducing new variables (so the original system is defined by a subset of the variables of a first order system) and (2) to consider only such systems that are locally solvable (Nixon 1991, [2] Olver 1986) because this can always be arranged, at least for linear systems, by adding extra equations by cross-differentiation provided the original system is consistent.

These techniques complement other techniques such as the use of symmetries that Olver has described [2] and should be applied first because of the drastic simplifications that can be obtained.

While developing these ideas, I was swapping between the general theory and the examples, each helping to improve the understanding of the other, and as a result it was difficult to find a way to present the work, with the options being to present the examples first and constantly refer forward to the general treatment, or the present the general theory first without prior motivating examples. In the end I chose the latter, so the outline of the paper is as follows for linear systems: In section 2 I show how to find the integrability conditions giving rise to local solvability. In section 3 I describe the extension of the method of minimising dimension to "partial" minimisation of dimension i.e. doing it for a subset of the system. In section 4 I describe another approach to the 2D Laplace equation in detail.

2 The integrability conditions and local solvability

Start with the linear system

$$\sum_{\imath=1}^{p}\sum_{l=1}^{n}\frac{\partial u_\imath}{\partial x_l}a_{\imath lk}(\boldsymbol{x})+\sum_{\imath=1}^{p}u_\imath b_{\imath k}(\boldsymbol{x})=0 \text{ for } 1\le k\le m. \tag{1}$$

Here I use $\imath$ where I used i before in (16) and following in [1] to distinguish it from the imaginary number i, and the 2D array called $\boldsymbol{a}$ there will be called $\boldsymbol{b}$ here to distinguish it from the 3D array also called $\boldsymbol{a}$ in [1].

Suppose the 3D array $a_{\imath jk}$ with dimensions $p\times n\times m$ is such that the $n\times n$ matrices

$$(c_\imath)_{jl}=\sum_{k=1}^{m}h_{kl}a_{\imath jk} \text{ are skew symmetric for } 1\le\imath\le p. \tag{2}$$

These are the equations derived by equating to zero all the second order terms arising from the linear combination

$$\sum_{k=1}^{m}\boldsymbol{h}_k(\boldsymbol{x}).\nabla\,(\text{ equation } k \text{ of } (1)) \tag{3}$$

so if the $\boldsymbol{h}_k$ satisfy (2), the resulting linear combination is also first order and can be written as

$$\sum_{\imath=1}^{p}\sum_{j=1}^{n}\frac{\partial u_i}{\partial x_j}\left[\sum_{k=1}^{m}\sum_{l=1}^{n}h_{kl}\frac{\partial a_{\imath jk}}{\partial x_l}+\sum_{k=1}^{m}b_{\imath k}h_{kj}\right]+\sum_{\imath=1}^{p}u_i\left[\sum_{k=1}^{m}\sum_{l=1}^{n}h_{kl}\frac{\partial b_{\imath k}}{\partial x_l}\right]=0. \tag{4}$$

The condition (2) can be written as

$$\sum_{k=1}^{m}a_{\imath jk}h_{kl}+\sum_{k=1}^{m}a_{\imath lk}h_{kj}=0 \text{ for } 1\le \imath\le p \text{ and } 1\le j,l\le n. \tag{5}$$

These are linear equations for the mn elements of $\boldsymbol{h}$ therefore they can be written in the form

$$\sum_{\beta}A_{\alpha\beta}H_{\beta}=0 \text{ for all } \alpha. \tag{6}$$

The parameter α is indexed by $\imath, j, l$ and β is indexed by the two indices of h, and the number of values of α and β are pn^2 and mn respectively (but note that the equations (2) have redundancy because if $l \neq j$ these indices can be swapped giving the same result).

The procedure I was suggesting in [1] to be applied to the system (1) is to repeatedly find $\boldsymbol{h}$, one set of all the coefficients h_{kl} satisfying (5) and then construct the corresponding equation (4) and add it to the original system (1). Keep doing this until there are no non-zero solutions for $\boldsymbol{h}$ such that the resulting PDE is not just linear combination of the original system (1), again of the form (1).

To understand equation (5) in terms of matrices let λ be an index that takes the place of l as an index of H. Then H and β will be indexed by k and λ, so that one can speak of row $(\imath, j, l)$ and column (k, λ) of A. To evaluate the element in this position, pick out the coefficient of $h_{k\lambda}$ in (5) giving $a_{\imath jk}\delta_{l\lambda}+a_{\imath lk}\delta_{j\lambda}$ making it obvious that A splits naturally into two terms which will be referred to as A_1 and A_2 respectively. If the indices l and λ vary most slowly in α and β respectively (and j varies slower than $\imath$) then A_1 can be written as an $n\times n$ matrix of matrices of dimension $pn\times m$ where the off-diagonal matrix terms of A_1 are zero and the diagonal terms are all the same, say B. Then B is the $pn\times m$ matrix having all the elements $a_{\imath jk}$ arranged so that this is the element on row $(\imath, j)$ and column k.

A_2 can also be split naturally into submatrices but is more complicated. First split B into n rows corresponding to the n values of j. These could be called $B_1 \dots B_n$ such that B_j is the $p\times m$ matrix with element $a_{\imath jk}$ on row $\imath$ and column k. Then A_2 is naturally split into an $n\times n$ matrix indexed by rows l and columns λ such that the (l, λ) element is a matrix like B but where all the sub-matrices B_j are replaced by zero except the one in the position of B_λ and it is B_l.

If this is sketched out it is obvious that all the rows of A_1 are replicated in the rows of A_2 and vice versa. Specifically, major row l subrow j (for all values of $\imath$) of A_1 are the same as major row j subrow l (for all values of $\imath$) of A_2 and they are given by the matrix B_j in position l amongst a set of n matrices all the same dimensions $(p \times m)$ in a row where these others are all zero matrices. Therefore each row of A is the sum of two rows of A_1. Also each row of A_1 will appear multiplied by two in A so every row of A_1 is a linear combination of the rows of A and vice versa i.e. the rows of A and A_1 span the same vector space. Considering A_1 shows that the dimension of this vector space is rn where r is the rank of B so the rank of A is mn if and only if the rank of B is m. In terms of these submatrices, (6) can be written as

$$B_l \tilde{\boldsymbol{h}}_j + B_j \tilde{\boldsymbol{h}}_l = 0 \text{ for } 1 \leq j, l \leq n \tag{7}$$

where $\tilde{\boldsymbol{h}}_j = \begin{pmatrix} h_{1j} \\ h_{2j} \\ \cdots \\ h_{mj} \end{pmatrix}$ i.e. the vectors $\tilde{\boldsymbol{h}}_j$ are the columns of the matrix $\boldsymbol{h}$ of which the rows are $\boldsymbol{h}_k$. The matrices B_j have dimensions $p \times m$ therefore if $m \leq p$ and if at least one of the B's eg B_s has full rank then (7) for $l = j = s$ gives $\tilde{\boldsymbol{h}}_s = 0$. Then put $l = s$ gives $B_s \tilde{\boldsymbol{h}}_j = 0$ for $1 \leq j \leq n$ i.e. $\boldsymbol{h} = 0$ so a condition under which this process of repeatedly adding integrability conditions to the original system stops is that at least one of the submatrices B_s has full rank because this ensures that any new integrability conditions obtained by this process are just $0 = 0$.

After one application of the method with just one non-zero solution for $\boldsymbol{h}$ picked out, the new array in place of $a_{\imath jk}$ is

$$a^*_{\imath jk} = \begin{cases} a_{\imath jk} \text{ for } 1 \leq k \leq m \\ e_{\imath j} \text{ for } k = m + 1 \end{cases} \tag{8}$$

where from (4)

$$e_{\imath j} = \sum_{k=1}^{m} \sum_{l=1}^{n} h_{kl} \frac{\partial a_{\imath jk}}{\partial x_l} + \sum_{k=1}^{m} b_{\imath k} h_{kj} \tag{9}$$

and the h_{kl} are the components of one non-trivial solution for $\boldsymbol{h}$. This would be expected to increase the rank of at least one of the matrices B_j by one. If it did not then there is a linear combination (LC) say w_k of the system (1) such that $\sum_{k=1}^{m} w_k a_{\imath jk} = e_{\imath j}$ and if from the newly derived equation (4) this linear combination of (1) is subtracted a linear combination of the $u_\imath = 0$ is obtained so if this is non-trivial, one of these variables can be eliminated reducing the size of the system and eventually this process will end as above.

In general at each step of the process there are 3 cases that can arise regardless of whether $m \leq p$ or not: (1) there is a non-zero solution $\boldsymbol{h}$ which leads to a new PDE (4) which is not a LC of the the system (1) (2) there is a non-zero solution $\boldsymbol{h}$ but it leads to a LC of (1) or (3) there is only the solution $\boldsymbol{h} = 0$. Case (1) cannot be repeated indefinitely unless an inconsistency arises showing that there are no solutions, because after some point, each time it happens more unknowns and their derivatives can be eliminated eventually leading to halting of the procedure i.e. cases (2) or (3) giving no new linearly independent equations.

Making a change of independent variables $x_1, \dots x_n \to t, y_1, \dots y_{n-1}$ gives

$$\frac{\partial u_\imath}{\partial x_l} = \sum_{j=1}^{n-1} \frac{\partial u_\imath}{\partial y_j} \frac{\partial y_j}{\partial x_l} + \frac{\partial u_\imath}{\partial t} \frac{\partial t}{\partial x_l} \tag{10}$$

and when expressed in terms of these variables, (1) gives

$$\sum_{\imath=1}^{p} \sum_{l=1}^{n} \left(\sum_{j=1}^{n-1} \frac{\partial u_\imath}{\partial y_j} \frac{\partial y_j}{\partial x_l} + \frac{\partial u_\imath}{\partial t} \frac{\partial t}{\partial x_l} \right) a_{\imath lk}(\boldsymbol{x}) + \sum_{\imath=1}^{p} u_\imath b_{\imath k}(\boldsymbol{x}) = 0 \text{ for } 1 \le k \le m. \tag{11}$$

These equations can be solved for $\frac{\partial u_\imath}{\partial t}$ if and only if there exists a vector $\frac{\partial t}{\partial x_l}$ such that

$$\operatorname{rank} \left(\sum_{l=1}^{n} \frac{\partial t}{\partial x_l} a_{\imath lk} \right) = \operatorname{rank} \left(\sum_{l=1}^{n} \frac{\partial t}{\partial x_l} a_{\imath lk}, \sum_{\imath=1}^{p} u_\imath b_{\imath k} \right) \tag{12}$$

where the matrices have row index $\imath$ and column index k and the one on the right is an augmented matrix with one extra column, but this is the condition that the original system can be put into Cauchy Kovalevskaya form ensuring that it is locally solvable ([2] p166). If $m = p$ the matrix M on the left is square and if at least one of the B_j has maximal rank then M is non-singular because this matrix is a linear combination of the B's it can be chosen to be non-singular because at least one of the B's is also. Therefore (12) holds and local solvability provided $m = p$ and at least one of the B_j has maximal rank.

3 Minimisation of dimension for a subset of the original system

The method used here is a modification and extension of the method I used in [1] to search for a reduced number of independent variables which can be used to express a linear system of PDE's. The modification is to only require a subset of $m' < m$ linear combinations of the original system but the enhanced requirement is that $r = 1$ i.e. the equations are reduced to ODE's. It is hoped that several such systems can be found that are together equivalent to the original system. Thus what is required is a change of independent variables from $x_1, \ldots x_n$ to $z_1, \ldots z_n$ such that say $z_1, \ldots z_{n-1}$ are absent from the derivatives of the linear combinations of original system when expressed in terms of the z's. Each such set of linear combinations of the original system forms a vector space at each point $\boldsymbol{x}$ and any set of spanning vectors can be chosen to represent it giving equivalent results. Following the notation of [1] with the modifications in section 2 above, if the linear combinations are

$$\sum_{k=1}^{m} g_{kq}(x) \left[\sum_{\imath=1}^{p} \sum_{j=1}^{n} \frac{\partial u_\imath}{\partial x_j} a_{\imath jk}(x) + \sum_{\imath=1}^{p} u_\imath b_{\imath k}(x) \right] = 0 \text{ for } 1 \leq q \leq m' \tag{13}$$

this gives the condition

$$\sum_{j=1}^{n} \sum_{k=1}^{m} g_{kq} \frac{\partial z_l}{\partial x_j} a_{\imath jk}(x) = 0 \text{ for } 1 \leq l \leq n-1; 1 \leq q \leq m'; 1 \leq \imath \leq p \tag{14}$$

i.e. for each q such that $1 \leq q \leq m'$, there are $n-1$ functionally independent solutions z of

$$\sum_{j=1}^{n} \sum_{k=1}^{m} g_{kq} \frac{\partial z}{\partial x_j} a_{\imath jk}(x) = 0 \text{ for } 1 \leq \imath \leq p \tag{15}$$

so the Lie algebra generated by $\boldsymbol{f}_{\imath q}$ for $1 \leq \imath \leq p$ where

$$(\boldsymbol{f}_{\imath q})_j = \sum_{j=1}^{n} \sum_{k=1}^{m} g_{kq} a_{\imath jk}(x) \frac{\partial}{\partial x_j} \tag{16}$$

has orbits of dimension 1 for each q satisfying $1 \leq q \leq m'$. This condition on the Lie algebra is trivial and only requires that the set of vector fields are all everywhere parallel i.e. the matrix representing them has rank 1 so for each q the condition is

$$\text{Rank} \left(\sum_{k=1}^{m} g_k a_{\imath jk}(x) \right) = 1. \tag{17}$$

This can be written a set of 2×2 determinants that are all zero i.e.

$$\left(\sum_{k_1=1}^{m} a_{11k_1} g_{k_1} \right) \left(\sum_{k_2=1}^{m} a_{\imath jk_2} g_{k_2} \right) - \left(\sum_{k_1=1}^{m} a_{1jk_1} g_{k_1} \right) \left(\sum_{k_2=1}^{m} a_{\imath 1k_2} g_{k_2} \right) = 0 \text{ for } 2 \leq j \leq n; 2 \leq \imath \leq p \tag{18}$$

or more compactly as

$$\sum_{k_1=1}^{m} \sum_{k_2=1}^{m} \begin{vmatrix} a_{11k_1} & a_{1jk_1} \\ a_{\imath 1k_2} & a_{\imath jk_2} \end{vmatrix} \frac{g_{k_1}}{g_m} \frac{g_{k_2}}{g_m} = 0 \text{ for } 2 \leq j \leq n; 2 \leq \imath \leq p \tag{19}$$

In the rest of this section it will be assumed that $m = p$ because this makes the results much simpler. The number of equations is $(n-1)(m-1)$ for the $m-1$ ratios of the g's so it is very overdetermined.

Each equation of (19) is a polynomial in the $m-1$ variables $z_k = g_k/g_m$ for $1 \leq k \leq m-1$ of total degree 2 equated to zero. Keeping j fixed and arbitrary, there are $m-1$ such equations. If from equations for $\imath$ is 2 and 3, the term for z_{m-1}^2 is eliminated by making a linear combination of these equations, the resulting equation will be linear in z_{m-1}, and can be solved for it. In more detail, if equation 2 in (19) is written as a polynomial in z_{m-1}, it has the form $P_{02}(z_1, \ldots z_{m-2}) + P_{12}(z_1 \ldots z_{m-2})z_{m-1} + P_{22}z_{m-1}^2 = 0$ where P_{02} has total degree 2, P_{12} has total degree 1 i.e. a linear combination of $z_1 \ldots z_{m-2}$, and P_{22} is a constant. This is because the whole expression has a total degree of 2 i.e. the sum of the powers of the z's in each term is ≤ 2. Solving for z_{m-1}^2 and substituting this into the corresponding equation for $\imath = 3$ with notation P_{03}, P_{13}, and P_{23} gives $z_{m-1} = \frac{P_{23}P_{02}-P_{22}P_{03}}{P_{22}P_{13}-P_{23}P_{12}}$ where the numerator has total degree 2 and the denominator has total degree 1. If this cancels down to another polynomial R linear in $z_1, \ldots z_{m-2}$ then substituting back gives another polynomial of total degree 2 in $z_1 \ldots z_{m-2}$ i.e. $P_{0i}(z_1 \ldots z_{m-2}) + P_{1i}(z_1 \ldots z_{m-2})R(z_1 \ldots z_{m-2}) + P_{2i}R(z_1 \ldots z_{m-2})^2 = 0$ and a linear combination of $z_1 \ldots z_{m-1}$ is zero. It is not clear under what circumstances this would be the case.

Go back to the case $m = 3$. Equation (19) can be written in the form

$$B_{0\imath} + B_{1\imath}z_1 + B_{2\imath}z_2 + B_{3\imath}z_1^2 + B_{4\imath}z_1z_2 + B_{5\imath}z_2^2 = 0 \tag{20}$$

where

$$\begin{aligned}
B_{0\imath} &= \begin{vmatrix} a_{113} & a_{1j3} \\ a_{\imath 13} & a_{\imath j3} \end{vmatrix} \\
B_{1\imath} &= \begin{vmatrix} a_{111} & a_{1j1} \\ a_{\imath 13} & a_{\imath j3} \end{vmatrix} + \begin{vmatrix} a_{113} & a_{1j3} \\ a_{\imath 11} & a_{\imath j1} \end{vmatrix} \\
B_{2\imath} &= \begin{vmatrix} a_{112} & a_{1j2} \\ a_{\imath 13} & a_{\imath j3} \end{vmatrix} + \begin{vmatrix} a_{113} & a_{1j3} \\ a_{\imath 12} & a_{\imath j2} \end{vmatrix} \\
B_{3\imath} &= \begin{vmatrix} a_{111} & a_{1j1} \\ a_{\imath 11} & a_{\imath j1} \end{vmatrix} \\
B_{4\imath} &= \begin{vmatrix} a_{111} & a_{1j1} \\ a_{\imath 12} & a_{\imath j2} \end{vmatrix} + \begin{vmatrix} a_{112} & a_{1j1} \\ a_{\imath 11} & a_{\imath j1} \end{vmatrix} \\
B_{5\imath} &= \begin{vmatrix} a_{112} & a_{1j2} \\ a_{\imath 12} & a_{\imath j2} \end{vmatrix}
\end{aligned} \tag{21}$$

The z_2^2 can be eliminated from these equations by taking B_{53} times (20) for $\imath = 2$ and subtracting B_{52} times (20) for $\imath = 3$. This can be written as

$$C_1 + C_2z_1 + C_3z_2 + C_4z_1^2 + C_5z_1z_2 = 0 \tag{22}$$

where

$$\begin{aligned} C_1 &= B_{53}B_{02} - B_{52}B_{03} = \begin{vmatrix} B_{53} & B_{52} \\ B_{03} & B_{02} \end{vmatrix} \\ C_2 &= B_{53}B_{12} - B_{52}B_{13} = \begin{vmatrix} B_{53} & B_{52} \\ B_{13} & B_{12} \end{vmatrix} \\ C_3 &= B_{53}B_{22} - B_{52}B_{23} = \begin{vmatrix} B_{53} & B_{52} \\ B_{23} & B_{22} \end{vmatrix} \\ C_4 &= B_{53}B_{32} - B_{52}B_{33} = \begin{vmatrix} B_{53} & B_{52} \\ B_{33} & B_{32} \end{vmatrix} \\ C_5 &= B_{53}B_{42} - B_{52}B_{43} = \begin{vmatrix} B_{53} & B_{52} \\ B_{43} & B_{42} \end{vmatrix} \end{aligned} \quad . \tag{23}$$

Solving for z_2 gives $z_2 = \frac{-C_1 - C_2 z_1 - C_4 z_1^2}{C_3 + C_5 z_1}$. If this cancels down to an expression linear in z_1 then the value of $z_1 = -C_3/C_5$ making the denominator vanish does the same for the numerator i.e. $C_1 - C_2\frac{C_3}{C_5} + C_4\frac{C_3^2}{C_5^2} = 0$ or

$$C_1C_5^2 - C_2C_3C_5 + C_4C_3^2 = 0 \tag{24}$$

If this happens, z_1 and z_2 are linearly related so (22) gives a quadratic equation for z_1 so there are in general two pairs of values of (z_1, z_2) so two 1D vector spaces of values of $\boldsymbol{g}$. Therefore there cannot be 3 different values of z_1 and z_2 which was expected by analogy with the examples in section (4) in two independent variables. This strongly suggests that this is a special degenerate case not generally to be expected.

The set of vectors $\boldsymbol{g}$ satisfying this could be just scalar multiples of each other or they could form the vector space of dimension ≥ 2 and there could in general be more than one such vector space satisfying this. Here the larger m' is, the larger the number of different vectors $\boldsymbol{g}$ that can be found is, the more useful the result is potentially.

4 Examples that motivated the general theory

4.1 The one dimensional wave equation

A generalised form of the 1D wave equation

$$\frac{\partial^2 u}{\partial x \partial y} + F\left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, u, x, y\right) = 0 \tag{25}$$

can be expressed as

$$\begin{aligned} &\tfrac{\partial u_1}{\partial y} + F\left(u_1, \tfrac{\partial u}{\partial y}, u, x, y\right) = 0 \\ &\tfrac{\partial u}{\partial x} - u_1 = 0 \end{aligned} \tag{26}$$

where if u is known over the whole space then u_1 can be obtained by integrating (26).1 along lines of constant x, and if u_1 is known over the whole space then u can be obtained by integrating (26).2 along lines of constant y. Thus the PDE is can be thought of as two coupled systems of ODE's. Such equations can also be written as Volterra integral equations.

4.2 The two dimensional Laplace equation

Another example follows where a similar thing happens is with the Laplace equation but is a much more complicated.

Suppose $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$ i.e. u satisfies the Laplace equation. In the notation of [1] let $x_1=x, x_2=y, u_1=u$, then the system of 3 equations is for example $k=1: u_2=\frac{\partial u_1}{\partial x_1}$, for $k=2: u_3=\frac{\partial u_1}{\partial x_2}$ which for $k=3: \frac{\partial u_2}{\partial x_1}+\frac{\partial u_3}{\partial x_2}=0$ and is a first order system of the form (1). The 3D array $a_{\imath jk}$ is given by

$$\boldsymbol{a}=\begin{pmatrix}1&0\\0&0\\0&0\end{pmatrix}\begin{pmatrix}0&1\\0&0\\0&0\end{pmatrix}\begin{pmatrix}0&0\\1&0\\0&1\end{pmatrix} \tag{27}$$

where the rows and columns of the separate matrices are indexed by $\imath$ and j respectively and k counts the matrices from left to right.

Equation (27) can be written as $a_{\imath jk}=\delta_{\imath 1}\delta_{j1}\delta_{k1}+\delta_{\imath 1}\delta_{j2}\delta_{k2}+(\delta_{\imath 2}\delta_{j1}+\delta_{\imath 3}\delta_{j2})\delta_{k3}$ and (2) gives $\sum_{k=1}^{3}h_{kl}a_{\imath jk}=h_{1l}\delta_{\imath 1}\delta_{j1}+h_{2l}\delta_{\imath 1}\delta_{j2}+h_{3l}(\delta_{\imath 2}\delta_{j1}+\delta_{\imath 3}\delta_{j2})$ is skew symmetric with respect to j and l for $1\le\imath\le 3$. For $j=l=1$ this gives $h_{11}\delta_{\imath 1}+h_{31}\delta_{\imath 2}=0$ i.e. $h_{11}=h_{31}=0$. For $j=l=2$ this gives similarly $h_{22}=h_{32}=0$. For $j=1,l=2$, equating it to minus the same expression with $j=2,l=1$ gives $h_{31}=h_{32}=0$ and $h_{12}=-h_{21}$. Only this last result allows for a non-zero solution and shows that the system must be extended by adding an extra equation from (3) in which the array $b_{\imath k}$ is the coefficient of $u_\imath$ in equation k i.e. $b=\begin{pmatrix}0&0&0\\1&0&0\\0&1&0\end{pmatrix}$ and $h_{kj}=\begin{pmatrix}0&-h_{21}\\h_{21}&0\\0&0\end{pmatrix}$ so $bh=\begin{pmatrix}0&0\\0&-h_{21}\\h_{21}&0\end{pmatrix}$. This becomes the fourth equation of the following system (28) the drastic simplification being due to the fact that all the coefficients are constants. Repeating this to search for a linear combination of the now four terms in (3) (because $m=4$) results in extra terms being added to (2) which are $h_{4l}(\delta_{\imath 2}\delta_{j2}-\delta_{\imath 1}\delta_{j3})$. These result only in the term $h_{42}\delta_{\imath 2}$ being added to the result for $j=l=2$ and $h_{41}\delta_{\imath 2}$ being added to the result for $j=2,l=1$ which show using $\imath = 2$ that $h_{41} = h_{42} = 0$.

$$\begin{cases} \frac{\partial u_1}{\partial x_1} - u_2 = 0 \\ \frac{\partial u_1}{\partial x_2} - u_3 = 0 \\ \frac{\partial u_2}{\partial x_1} + \frac{\partial u_3}{\partial x_2} = 0 \\ \frac{\partial u_2}{\partial x_2} - \frac{\partial u_3}{\partial x_1} = 0 \end{cases} . \tag{28}$$

Therefore there are no new results derivable from the system by one application of the procedure indicated in [1] pages 2920-2921. Notice that in (28), $p = 3$ and $m = 4$ so this is an overdetermined system but the above procedure has come to an and there is no apparent inconsistency so there is something unexpected happening in this system. The $a_{\imath jk}$ are now given by the following 4 matrices indexed by row $\imath$ and column j in each matrix k from 1 to 4:

$$\boldsymbol{a} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{29}$$

Applying the next step of the analysis procedure proposed in [1] which is to minimise the dimension of the space in which the equations can be expressed gives that because all the coefficients $a_{\imath jk}$ are constant, all the Lie brackets (commutators) of the differential operators $\sum_{j=1}^{n} a_{\imath jk} \frac{\partial}{\partial x_j}$ (indexed by $\imath$ and k) are zero and they span 2-dimensional space, there is no reduction of dimension for independent variables. Looking for a reduced number of dependent variables gives [1] eq 32 i.e. $\sum_{\imath=1}^{p} d_\imath a_{\imath\{jk\}} = 0$ so the rank of $a_{\imath jk}$ has to be analysed after stacking the sub-matrices another way by combining the indices j and k giving rank $= 3$ so there is no reduction and there are 3 dependent variables.

Applying "partial" minimisation of dimension, from (29) which is $a_{\imath jk}$, the matrix in (17) must be constructed thus

$$\begin{pmatrix} g_1 & g_2 \\ g_3 & g_4 \\ -g_4 & g_3 \end{pmatrix} . \tag{30}$$

The condition that this has rank 1 determines conditions on linear combinations $\boldsymbol{g}$ of the system giving $g_1 g_4 = g_2 g_3, g_1 g_3 = -g_4 g_2, g_3^2 = -g_4^2$ so ($g_4 = ig_3$ and $g_2 = ig_1$) or ($g_4 = -ig_3$ and $g_2 = -ig_1$). Therefore the matrix has the two forms so in this example $m' = 2$.

$$\begin{pmatrix} g_1 & ig_1 \\ g_3 & ig_3 \\ -ig_3 & g_3 \end{pmatrix} \tag{31}$$

or

$$\begin{pmatrix} g_1 & -ig_1 \\ g_3 & -ig_3 \\ ig_3 & g_3 \end{pmatrix} \tag{32}$$

corresponding to $\boldsymbol{g} = (g_1, ig_1, g_3, ig_3)$ and $(g_1, -ig_1, g_3, -ig_3)$ respectively. The first of these gives two independent results corresponding to $\boldsymbol{g} = (1, i, 0, 0)$ and $\boldsymbol{g} = (0, 0, 1, i)$ giving matrices proportional to $(1, 0, 0)^T$ and $(0, 1, 1)^T$ respectively. Similarly for the second case $\boldsymbol{g} = (1, -i, 0, 0)$ and $\boldsymbol{g} = (0, 0, 1, -i)$ gives rise to rank 1 matrices proportional to $(1, 0, 0)^T$ and $(0, 1, 1)^T$ respectively. However linear combinations of $\boldsymbol{g} = (1, i, 0, 0)$ and $\boldsymbol{g} = (1, -i, 0, 0)$ also give results proportional to $(1, 0, 0)^T$ but linear combinations of $\boldsymbol{g} = (0, 0, 1, i)$ and $\boldsymbol{g} = (0, 0, 1, -i)$ do *not* give rise to a matrix of rank 1 unless one of the coefficients is zero. This represents some complex structure implicit in (29) that relates to the equivalent form presented below.

In the first case, there are two linear combinations that have $r = 1$:

$$\begin{aligned} u_{1,1} - u_2 + i(u_{1,2} - u_3) = 0 \\ u_{2,1} + u_{3,2} + i(u_{2,2} - u_{3,1}) = 0 \end{aligned} \tag{33}$$

using the abbreviation $z_{,j}$ for $\frac{\partial z}{\partial x_j}$ for any variable z which can be written as

$$\begin{aligned} &\tfrac{du_1}{ds_1} - u_2 - iu_3 = 0 \\ &\tfrac{du_2}{ds_1} - i\tfrac{du_3}{ds_1} = 0 \end{aligned} \tag{34}$$

where

$$\frac{d}{ds_1} = \frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2} \tag{35}$$

to make its 1-dimensional status clear. The first of (33) can be integrated to get u_1 in terms of u_2 and u_3 while the second can be integrated to get u_2 in terms of u_3 (or vice versa) along the same set of curves. Repeating this for the second solution is just the same as (33) with i replaced by $-i$, in another set of curves (actually in both cases they are straight lines in complex space)

$$\begin{aligned} &\tfrac{du_1}{ds_2} - u_2 + iu_3 = 0 \\ &\tfrac{du_2}{ds_2} + i\tfrac{du_3}{ds_2} = 0 \end{aligned} \tag{36}$$

where

$$\frac{d}{ds_2} = \frac{\partial}{\partial x_1} - i\frac{\partial}{\partial x_2} \tag{37}$$

so there are in total 4 linear combinations of the equations (28) which together span the complete 4 dimensional space of possible linear combinations of the 4 equations in (28). This is an equivalent way of expressing the system likely to be useful theoretically in deriving properties of the solution and possibly finding it numerically from appropriate boundary conditions by integrating the equations as indicated and using an iterative method for solving them starting from an initial estimate which is constant along a set of curves that intersect the initial surface once each. The obvious choice of coordinate system is now such that s_2 constant in (35) and s_1 is constant in (37). Then the equations (34) and (36) can be written as

$$\left.\begin{aligned}\tfrac{\partial u_1}{\partial s_1} - u_2 - iu_3 = 0\\ \tfrac{\partial u_2}{\partial s_1} - i\tfrac{\partial u_3}{\partial s_1} = 0\end{aligned}\right\} \quad \left.\begin{aligned}\tfrac{\partial u_1}{\partial s_2} - u_2 + iu_3 = 0\\ \tfrac{\partial u_2}{\partial s_2} + i\tfrac{\partial u_3}{\partial s_2} = 0\end{aligned}\right\} \tag{38}$$

The first system with independent variable s_1 can be integrated along lines of constant s_2 while the second system with independent variable s_2 can be integrated along lines of constant s_1, and both systems must be solved simultaneously e.g. by iteration. The coordinate system requires the operators in (35) and (37) to commute which is obviously the case and (37) gives s_1 is constant or $0 = \frac{ds_1}{ds_2} = \frac{\partial s_1}{\partial x_1} + \frac{\partial s_1}{\partial x_2}\frac{dx_2}{dx_1}$ on lines given by $\frac{dx_2}{dx_1} = -i$ i.e. $s_1 = x_2 + ix_1$. Similarly $s_2 = x_2 - ix_1$.

For these equations $m = 4$, $n = 2$ and $p = 3$ and if the procedure in section 2 is applied again it shows that the matrices B_j are

$$B_1 = \begin{pmatrix} 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & -i & 0 & 0 \end{pmatrix}, B_2 = \begin{pmatrix} 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & 0 & i \end{pmatrix}, \tag{39}$$

both having rank 2. The equations (7) reduce to $h_{11} = h_{21} = h_{41} = h_{22} = h_{32} = h_{42} = 0$ and $h_{12} + h_{31} = 0$. Therefore the only equation arising from (3) is

$$\frac{\partial}{\partial s_2}\left(\frac{\partial u_1}{\partial s_1} - u_2 - iu_3\right) - \frac{\partial}{\partial s_1}\left(\frac{\partial u_1}{\partial s_2} - u_2 + iu_3\right) = 0 \tag{40}$$

which reduces to a LC of (38).2 and (38).4 so this is an over-determined system such that when the procedure above is applied it does not lead to an inconsistency.

The equations (38) can be expressed only in terms of real variables by splitting up the complex variables u_1, u_2, u_3 into real and imaginary parts but this seems to serve no purpose, just doubling the number of independent and dependent variables.

The general solution of the system (38) can be obtained by integrating (38).2 and (38).4 respectively giving

$$\begin{aligned} u_2 - iu_3 &= f(s_2)\\ u_2 + iu_3 &= g(s_1) \end{aligned} \tag{41}$$

where $f()$ and $g()$ are arbitrary functions, so

$$\begin{array}{l} u_2 = \frac{1}{2}(f(s_2) + g(s_1)) \\ u_3 = \frac{1}{2i}(-f(s_2) + g(s_1)) \end{array} . \tag{42}$$

Then equation (38).3 integrates to $u_1 = \int^{s_2} dt f(t) + h(s_1)$ where $h()$ is also an arbitrary function. This is consistent with equation (38).1 which integrates to $u_1 = \int^{s_1} dt g(t) + q(s_2)$ where $q()$ is also arbitrary and both together give

$$u(s_1, s_2) = h(s_1) + q(s_2) = h(x_2 + ix_1) + q(x_2 - ix_1). \tag{43}$$

u should be real valued and x_1 and x_2 are real. Suppose u is real valued on the real line, then $h(x_2) + q(x_2)$ is real. This makes $h()$ and $q()$ functions of a complex variable $z = x_2 + ix_1$. Clearly there must be some restriction on h and q because otherwise put eg $q = 0$ then $u = h(z)$ can equal any function of $\Re(z)$ and $\Im(z)$ which this makes the solution quite arbitrary. If $h()$ and $q()$ are analytic this problem is solved.

It is important to note that the CK theorem does not apply to this case because the original equation the Laplace equation is elliptic.

Do a more complex example where a_{ijk} are not constants.

(on this website www.bluesky-home.co.uk)

************ beyond here is old work - needs looking at again

5 Non-linear systems

In order to find first order PDEs for $\boldsymbol{u}$ derivable from

$$F_k(\boldsymbol{x}, \boldsymbol{u}, \nabla \boldsymbol{u}) = 0 \text{ for } 1 \leq k \leq m \tag{44}$$

I look for linear combinations, with coefficients that can depend on $(\boldsymbol{x}, \boldsymbol{u}, \nabla \boldsymbol{u}) = (\boldsymbol{x}, u^{(1)})$, of all the first total (with respect to x_i regarding $\boldsymbol{u}$ as fixed unknown functions of $\boldsymbol{x}$) derivatives of the F_k that do not involve second derivatives of $\boldsymbol{u}$. Linear combinations are considered because the second derivatives occur only linearly in the first total derivatives. i.e. consider expressions

$$\sum_{k=1}^{m} \sum_{l=1}^{n} \alpha_{kl}(x, \boldsymbol{u}^{(1)}) \frac{dF_k}{dx_l} \tag{45}$$

that are independent of second derivatives of $\boldsymbol{u}$. This results in equations corresponding to (37) for the α coefficients i.e.

$$\sum_{k=1}^{m} \left[\alpha_{kl}(\boldsymbol{x}, \boldsymbol{u}^{(1)}) \frac{\partial F_k}{\partial u_{i,j}} + \alpha_{kj}(\boldsymbol{x}, \boldsymbol{u}^{(1)}) \frac{\partial F_k}{\partial u_{i,l}} \right] = 0 \text{ for } 1 \leq j \leq l \leq n \text{ and } 1 \leq i \leq p. \tag{46}$$

Then for each linearly independent set of α satisfying this, the equation sought derived from the original system (44) is

$$\sum_{k=1}^{m}\sum_{l=1}^{n}\alpha_{kl}(\boldsymbol{x},\boldsymbol{u}^{(1)})\frac{dF_k}{dx_l}=0 \tag{47}$$

i.e. (45) equated to zero because the F_k and their total derivatives are all identically zero for all solutions of the original system (44). Then a complete set of linearly independent α satisfying the linear equation (46) will provide a complete set of independent first order equations (47) derivable from the original system by taking first derivatives only. Because there are $pn(n+1)/2$ equations for mn coefficents α_{kl}, the case when the number of equations equals the number of unknowns i.e. $m=p$ implies that the number of equations for the α exceeeds the number of α coeffients except when $n=1$, thus if there is more than one independent variable the equations (46) typically have only the trivial solutions i.e. there are no extra integrability conditions. The derivatives $\frac{\partial F_k}{\partial u_{i,j}}$ can in any example be evaluated as functions of $\boldsymbol{x},\boldsymbol{u},\nabla\boldsymbol{u}$ have to be treated as fixed constants in equations (46) and this system must be solved for each point $(\boldsymbol{x},\boldsymbol{u},\nabla\boldsymbol{u})$ separately.

However note that some linear combinations of the α can correspond to some of the existing F_k, so it is not necessarily true that the number of linearly independent solutions (A) for α is the number of independent *new* derived F_k (B) to added to the system as the result of one step of the completion procedure, in general $A\geq B$. Example (27) of my paper with $p=m=n=2$ shows this. Here $a_{111}=1$, $a_{211}=1$, $a_{122}=1$, $a_{222}=1$ and all the other $a_{ijk}=0$. The first completion step leads to $\alpha_{11}=\alpha_{22}=0$ and $\alpha_{12}=-\alpha_{21}$, and the new equation

$$F_3=-u_{1,2}+u_{2,2}+u_{1,1}-u_{2,1}=0. \tag{48}$$

The second completion step (now with $n=p=2$, and $m=3$) starts with the extended 3D array of coefficients having the extra values

$$\begin{array}{ll} a_{113}=1 & a_{213}=-1 \\ a_{123}=-1 & a_{223}=1 \end{array} \tag{49}$$

and gives the second set of equations for the α which reduce to

$$\begin{array}{l} \alpha_{11}=\alpha_{31}=0 \\ \alpha_{12}+\alpha_{21}=0 \\ \alpha_{32}=\alpha_{22}=0. \end{array} \tag{50}$$

This leads to no more solutions ie. no solutions that are not already equations included in the system.

(A) an example where no extra integrability conditions were needed (B and C) An example where this has to be repeated to get all the integrability conditions. Suppose the original system is

$$\begin{aligned} F_1 &\equiv (u_{1,1})^2 + u_1 u_{1,2} = 0 \\ F_2 &\equiv u_1 + x_1 u_{1,2} = 0 \end{aligned} \tag{51}$$

Then

$$\begin{aligned} \tfrac{\partial F_1}{\partial u_{1,1}} &= 2u_{1,1} \\ \tfrac{\partial F_1}{\partial u_{1,2}} &= u_1 \\ \tfrac{\partial F_2}{\partial u_{1,1}} &= 0 \\ \tfrac{\partial F_2}{\partial u_{1,2}} &= x_1 \end{aligned} \tag{52}$$

Then there are just three equations for the α having $i = 1$ and respectively $(j, l) = (1, 1), (1, 2), (2, 2)$. These are easily shown to be

$$\begin{aligned} &\alpha_{1,1} = 0 \\ &2u_{1,1}\alpha_{1,2} + \alpha_{2,1}x_1 = 0 \\ &\alpha_{1,2}u_1 + \alpha_{2,2}x_1 = 0 \end{aligned} \tag{53}$$

from which there is one linearly independent solution

$$\boldsymbol{\alpha} \equiv (\alpha_{11}, \alpha_{12}, \alpha_{21}, \alpha_{22}) = \left(0, -\frac{x_1}{u_1}\alpha_{22}, \frac{2u_{1,1}}{u_1}\alpha_{22}, \alpha_{22}\right). \tag{54}$$

This, from (47), gives rise to the new derived equation which simplifies to

$$F_3 \equiv -\frac{x_1}{u_1}(u_{1,2})^2 + \frac{2u_{1,1}^2}{u_1} + 2\frac{u_{1,1}u_{1,2}}{u_1} + u_{1,2} = 0 \tag{55}$$

Now the system has $m = 3$, and when this procedure is repeated, there are three more independent linear combinations of the total derivatives of F_1, F_2, F_3 that are zero. There is clearly no point in going any further because from (51) it follows that $F_3 = -2u_1(x_1)^{-3/2}$ and equating it to zero gives $u_1 = 0$ as the only solution when $x_1 \neq 0$. This trivial example does however show that more than one round of finding extra integrability conditions by this method in general gives new results, and it should in general be repeated until no new results are obtained.

(D) Searching for new independent variables to get reduced dimension $r = 1$ When searching for linear combinations that reduce the number of independent variables, allow the coefficients to be dependent on $\boldsymbol{x}$ and $u^{(1)}$. Should combine searching for new independent and dependent variables.

If for example $r = 1$, this is not much use unless $p = 1$.

The use of the Cauchy Riemann equations and analytic transformations. Suppose $u + iv$ is an analytic function of $x + iy$. Then the Cauchy Riemann equations are

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \tag{56}$$

Now suppose

$$z = f_1(u, v). \tag{57}$$

Then

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} - \frac{\partial z}{\partial v}\frac{\partial u}{\partial y} \tag{58}$$

and

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial u}{\partial x} \tag{59}$$

The CR equations are formally reducible to $r = 1$, but the coefficients are imaginary. Do in detail

For the general case where no simplifications are possible, one would expect that an iterative method leading to simultaneous convergence of $u_1 \ldots u_p$ starting from initial estimates that agree with the boundary conditions could be used. If $p = m$ this could be implemented by assigning each equation to a member of the set of dependent variables. Then starting from their initial estimates, use each equation to update the associated variable using the current values for all the other ones. This could be done by direct integration along characteristics because the equation is first order in the single unknown being calculated, and Monge's method applies. By repeated cycling round doing this for all the equations would give a scheme that could converge. This is what I would call the 'integrate and iterate' strategy i.e. convert the ODE's system to integral equations of Volterra type by integration of both sides. Update each variable in turn to hopefully get a set of such equations that can be solved by iteration to convergence establishing existence and uniqueness of the solution. This is a known general strategy to in principle solve a very wide class of differential equations giving unique solutions. This should be seen in its most general form. Need to show (1) The above method converges if sufficiently good initial estimates are given. Then the solution converged to satisfies the system of PDEs. Start by doing this for ODEs.

This establishes that the solution exists in a region. It may allow uniqueness to be shown, or use the Cauchy Kovalevskaya theorem. The region in which the solution is obtained is obviously the intersection of the regions determined by the m sets of characteristics.

Show that the singular cases when some sets of characteristics coincide correspond to the cases when minimisation of dimension leads to a lower dimensional problem as defined above.

$\frac{\partial F_k}{\partial u_{i,j}}$ as directions for integration of Monge Equations for u_i assuming all the other $\boldsymbol{u}$ are their final values in an iteration.

C.K. theorem for first order systems: (adaptation of argument in P.Olver)

Suppose the transformed system is written with one special independent variable, which I call t, and the others $y_1 \dots y_{n-1}$,

$$F_k^*(t, y_1, \dots y_{n-1}, \boldsymbol{u}^{(1)}) = 0 \text{ for } 1 \le k \le m. \tag{60}$$

This can be solved for derivatives with respect to t i.e.

$$\frac{\partial u_k}{\partial t} = H_k\left(t, y_1, \dots y_{n-1}, \frac{\partial \boldsymbol{u}}{\partial y_1}, \dots \frac{\partial \boldsymbol{u}}{\partial y_{n-1}}\right) \text{ for } 1 \le k \le m. \tag{61}$$

for which the C.K. theorem applies iff

$$\det\left(\frac{\partial F_k^*}{\partial(\frac{\partial \boldsymbol{u}}{\partial t})}\right) \neq 0 \tag{62}$$

Using the change of variable, this matrix can be written as follows

$$\frac{\partial F_k^*}{\partial\left(\frac{\partial u_i}{\partial t}\right)} = \sum_l \frac{\partial F_k}{\partial u_{i,l}} \frac{\partial u_{i,l}}{\partial\left(\frac{\partial u_i}{\partial t}\right)} \tag{63}$$

where using the chain rule for the change of variables, the last term simplifies to $\frac{\partial t}{\partial x_l}$.

Therefore the C.K. theorem applies giving a unique solution to the boundary value problem iff

$$\det\left(\sum_{l=1}^{n} \frac{\partial F_k}{\partial u_{i,l}} w_l\right) \neq 0 \tag{64}$$

where $\boldsymbol{w} = \nabla t$ is perpendicular to the initial surface. This condition can be written as

$$\sum_p \prod_{k=1}^{m} \left(\sum_{l_k=1}^{n} \frac{\partial F_k}{\partial u_{p(k),l_k}} w_k\right) (-1)^{\mathrm{sign}(p)}. \tag{65}$$

This can be further expanded as a condition involving a polynomial in the components $\boldsymbol{w}$ as follows

$$\sum_{l_1=1}^{n} \dots \sum_{l_m=1}^{n} w_{l_1} \dots w_{l_m} \left(\sum_p (-1)^{\mathrm{sign}(p)} \frac{\partial F_1}{\partial u_{p(1),l_1}} \dots \frac{\partial F_m}{\partial u_{p(m),l_m}}\right) = 0 \tag{66}$$

From this it follows that after fixing $w_1, \dots w_{n-1}$, this is a single algebraic equation for w_n of degree m with coefficients that depend on $\frac{\partial F_k}{\partial u_{i,l}}$, so this has up to m real solutions and most vectors w don't satisfy this condition. The vectors w satisfying this clearly are not typically a linear space but each w can always be multiplied by a scalar to get another solution, so I call the solutions sets at each point cones. For the nonlinear case, the cone will depend on the unknown $\boldsymbol{u}$ itself, and even in the linear case it will in general be dependent on $\boldsymbol{x}$.

Are integrability conditions a consequence of this degeneracy too?

If $m > p$ in general there will not be a solution to the boundary value problem. Solutions with smaller dimensional boundaries might be possible.

About the simplest possible example is the Cauchy Riemann equations which shows that this simple idea can run into problems. Use equation 1 to calculate u from v by integrating along lines parallel to the x axis, or calculate v from u by integrating ... y axis. Using equation 2 calculate u from v integrate parallel to the y axis 2 calculate v from u integrate parallel to the x axis

The problem here is that the calculation $u \to v$ by equation (1) and the calculation $v \to u$ by equation (2) give characteristics that are parallel, and likewise for the other way round.

For minimisation of the number of independent variables resulting in (55) and (56) to r, the three-dimensional array G with dimensions $m \times p \times n$ must satisfy

$$\operatorname{Rank}\left(M_{ij} = \sum_{k=1}^{m} G_{kij} h_k\right) = r \tag{67}$$

for some h. Requiring $r = 1$ for just one linear combination h would impose $pn - p - n + 1$ conditions on M because $n + p + 1$ parameters uniquely specify a $p \times n$ matrix of rank 1. Therefore there will not in general be a reduction to $r = 1$ for even a single linear combination of the equations. However if it does happen for a single linear combination, the equations (55) (n) and the second member of (56) after eliminating b_{ij} (p) can be in principle be integrated from the boundary (if appropriate boundary conditions are given) and allow one of the unknowns to be eliminated by back-substitution of the result into the original system.

Show how the new variables are constructed to do this.

Given a system of PDE and a surface S, what conditions are imposed by the system within S? Possible answers: (1) no condition i.e. $\boldsymbol{u}$ is unrestricted (2) There are no solutions for $\boldsymbol{u}$ so any conditions could be added (vacuously).(3) There could be some conditions.

This question is answered by the minimisation of dimension argument, because such restrictions, if present, would be systems of PDEs defined within S. So if a minimised dimension result was found within S, it must be included, and S is a characteristic surface, otherwise none are possible, and S is a non-characteristic surface.

Dependence of r on $\boldsymbol{x}$. It is not necessarily true that regions of different

r (dimension of the Lie algebra generated by the $\boldsymbol{f_i}$ at a point) correspond to integral surfaces. If a region of dimension $< n$ with tangent space X a point is identified having a given value of r for a L.C. of the system, the space Y spanned by the L.I. vectors from the completion of the $\boldsymbol{f_i}$ does not necessarily relate to X. If $Y \subseteq X$ it is possible to write an equation in the reduced dimension in a nbd. of the point. Otherwise it won't work. This is another integrability condition I. In general the partial function

$$r(\boldsymbol{x}) = \begin{cases} r \text{ if } I(\boldsymbol{x}) \\ \text{undefined if } \tilde{I}(\boldsymbol{x}) \end{cases} \tag{68}$$

detefines the effective reduced dimension for a L.C. of the original system.

From the general theory point of view, thinking of any variables complex is unnecessary because any such system can be written using the real and imaginary parts separately and adding the Cauchy Riemann equations if analytic solutions are wanted. Thus the new system equivalent to the system where complex or analytic solutions is sought, has only real solutions, coefficients and boundary conditions only. Indeed it would be possible to take the resulting system defined above and intepret the variables as complex and again represent everything in terms of real variables. This is clearly unnecessary "complexification".

The general problem for determined systems ($p = m$) is to establish existence and uniqueness of solutions in a region containing the initial data. The basic theorem for this is the Cauchy Kovalevskaya (CK) theorem. One way to imagine how the solution might be constructed is to start with initial estimates of $u_1, \ldots u_m$ consistent with the initial data then sequentially update $u_1, u_2, \ldots u_m$. For each of these m updates one equation of the system is used, so that they are each used once. Thus each equation of the system is a single first order PDE for the single unknown with all the other ones replaced by their current values. Thus Monge's method of integral strips applies. Once a variable has been updated, this updated value is used in all subsequent calculations until it is updated again. The cycle of updates should be repeated to convergence to any desired degree of accuracy. G_{kij} is the direction vector with components indexed by j for the propagation of u_i obtained from equation k, i.e. m^2 vectors in n dimensions. Attempts to prove that this converges in analogy with the Picard theorem for ODE's failed because it is nor clear what analogue of the Lipschitz condition or metric could be used that would give rise to a unique fixed point. The problem comes from the derivatives of all the variables in the equations.

With the CK theorem in mind, reductions of dimension that are possible should be seen as singular cases that are exceptional and if they occur, special techniques are needed (minimisation of dimension together with adding in the integrability conditions).

It is possible that different conditions occur in different regions of the space of independent variables. For the CK theorem I think no partial reductions of dimension should be possible on the initial surface i.e. all the directions for integrating each variable (at convergence i.e. when all the variables are known) lead out of the initial surface. Were they not to do so the initial surface could not have independently defined data, i.e. some extra conditions would have to be satisfied.

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